Oxford Cambridge and RSA

Monday 18 October 2021 – Afternoon

A Level Mathematics A

H240/03 Pure Mathematics and Mechanics

Time allowed: 2 hours

You must have:
- the Printed Answer Booklet
- a scientific or graphical calculator

INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. You can use extra paper if you need to, but you must clearly show your candidate number, the centre number and the question numbers.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give non-exact numerical answers correct to **3** significant figures unless a different degree of accuracy is specified in the question.
- The acceleration due to gravity is denoted by $g\,\mathrm{m\,s^{-2}}$. When a numerical value is needed use $g = 9.8$ unless a different value is specified in the question.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

INFORMATION
- The total mark for this paper is **100**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **12** pages.

ADVICE
- Read each question carefully before you start your answer.

Formulae
A Level Mathematics A (H240)

Arithmetic series

$S_n = \frac{1}{2}n(a+l) = \frac{1}{2}n\{2a+(n-1)d\}$

Geometric series

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_\infty = \frac{a}{1-r} \quad \text{for } |r| < 1$$

Binomial series

$$(a+b)^n = a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + \ldots + {}^nC_r a^{n-r}b^r + \ldots + b^n \qquad (n \in \mathbb{N}),$$

where ${}^nC_r = {}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots \qquad (|x| < 1,\ n \in \mathbb{R})$$

Differentiation

$\mathrm{f}(x)$	$\mathrm{f}'(x)$
$\tan kx$	$k\sec^2 kx$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$

Quotient rule $y = \frac{u}{v}$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$

Differentiation from first principles

$$\mathrm{f}'(x) = \lim_{h \to 0} \frac{\mathrm{f}(x+h) - \mathrm{f}(x)}{h}$$

Integration

$$\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\,\mathrm{d}x = \ln|\mathrm{f}(x)| + c$$

$$\int \mathrm{f}'(x)\big(\mathrm{f}(x)\big)^n\,\mathrm{d}x = \frac{1}{n+1}\big(\mathrm{f}(x)\big)^{n+1} + c$$

Integration by parts $\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$

Small angle approximations

$\sin\theta \approx \theta$, $\cos\theta \approx 1 - \frac{1}{2}\theta^2$, $\tan\theta \approx \theta$ where θ is measured in radians

Trigonometric identities

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \qquad \left(A \pm B \neq (k + \tfrac{1}{2})\pi\right)$

Numerical methods

Trapezium rule: $\int_a^b y\,\mathrm{d}x \approx \frac{1}{2}h\{(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\}$, where $h = \dfrac{b-a}{n}$

The Newton-Raphson iteration for solving $\mathrm{f}(x) = 0$: $x_{n+1} = x_n - \dfrac{\mathrm{f}(x_n)}{\mathrm{f}'(x_n)}$

Probability

$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B \mid A) = \mathrm{P}(B)\mathrm{P}(A \mid B)$ **or** $\mathrm{P}(A \mid B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$

Standard deviation

$\sqrt{\dfrac{\Sigma(x-\bar{x})^2}{n}} = \sqrt{\dfrac{\Sigma x^2}{n} - \bar{x}^2}$ **or** $\sqrt{\dfrac{\Sigma f(x-\bar{x})^2}{\Sigma f}} = \sqrt{\dfrac{\Sigma fx^2}{\Sigma f} - \bar{x}^2}$

The binomial distribution

If $X \sim \mathrm{B}(n, p)$ then $\mathrm{P}(X = x) = \dbinom{n}{x} p^x (1-p)^{n-x}$, mean of X is np, variance of X is $np(1-p)$

Hypothesis test for the mean of a normal distribution

If $X \sim \mathrm{N}(\mu, \sigma^2)$ then $\bar{X} \sim \mathrm{N}\left(\mu, \dfrac{\sigma^2}{n}\right)$ and $\dfrac{\bar{X} - \mu}{\sigma / \sqrt{n}} \sim \mathrm{N}(0, 1)$

Percentage points of the normal distribution

If Z has a normal distribution with mean 0 and variance 1 then, for each value of p, the table gives the value of z such that $\mathrm{P}(Z \leqslant z) = p$.

p	0.75	0.90	0.95	0.975	0.99	0.995	0.9975	0.999	0.9995
z	0.674	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291

Kinematics

Motion in a straight line

$v = u + at$

$s = ut + \frac{1}{2}at^2$

$s = \frac{1}{2}(u + v)t$

$v^2 = u^2 + 2as$

$s = vt - \frac{1}{2}at^2$

Motion in two dimensions

$\mathbf{v} = \mathbf{u} + \mathbf{a}t$

$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$

$\mathbf{s} = \frac{1}{2}(\mathbf{u} + \mathbf{v})t$

$\mathbf{s} = \mathbf{v}t - \frac{1}{2}\mathbf{a}t^2$

Section A: Pure Mathematics

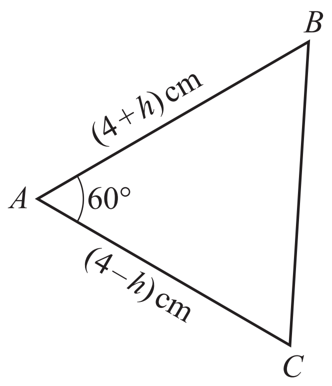
Answer **all** the questions.

1 Show in a sketch the region of the x-y plane within which all three of the following inequalities hold.

$y \geqslant x^2, \quad x+y \leqslant 2, \quad x \geqslant 0.$

You should indicate the region for which the inequalities hold by labelling the region R. **[3]**

2



The diagram shows triangle ABC in which angle A is 60° and the lengths of AB and AC are $(4+h)$ cm and $(4-h)$ cm respectively.

(a) Show that the length of BC is p cm where

$p^2 = 16+3h^2.$ **[2]**

(b) Hence show that, when h is small, $p \approx 4+\lambda h^2+\mu h^4$, where λ and μ are rational numbers whose values are to be determined. **[4]**

3 An arithmetic progression has first term 2 and common difference d, where $d \neq 0$. The first, third and thirteenth terms of this progression are also the first, second and third terms, respectively, of a geometric progression.

By determining d, show that the arithmetic progression is an increasing sequence. **[5]**

4 **(a)** Sketch, on a single diagram, the following graphs.

- $y = |x - 1|$
- $y = \frac{k}{x}$, where k is a negative constant **[2]**

(b) Hence explain why the equation $x|x-1| = k$ has exactly one real root for any negative value of k. **[1]**

(c) Determine the real root of the equation $x|x-1| = -6$. **[2]**

5 A particle P moves along a straight line in such a way that at time t seconds P has velocity $v\,\mathrm{m\,s^{-1}}$, where

$$v = 12\cos t + 5\sin t.$$

(a) Express v in the form $R\cos(t - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$. Give the value of α correct to **4** significant figures. **[3]**

(b) Hence find the two smallest positive values of t for which P is moving, in either direction, with a speed of $3\,\mathrm{m\,s^{-1}}$. **[3]**

6 The equation $6\arcsin(2x - 1) - x^2 = 0$ has exactly one real root.

(a) Show by calculation that the root lies between 0.5 and 0.6. **[2]**

In order to find the root, the iterative formula

$$x_{n+1} = p + q\sin(rx_n^2),$$

with initial value $x_0 = 0.5$, is to be used.

(b) Determine the values of the constants p, q and r. **[2]**

(c) Hence find the root correct to **4** significant figures. Show the result of each step of the iteration process. **[2]**

7 A curve C in the x-y plane has the property that the gradient of the tangent at the point $P(x, y)$ is three times the gradient of the line joining the point (3, 2) to P.

(a) Express this property in the form of a differential equation. **[2]**

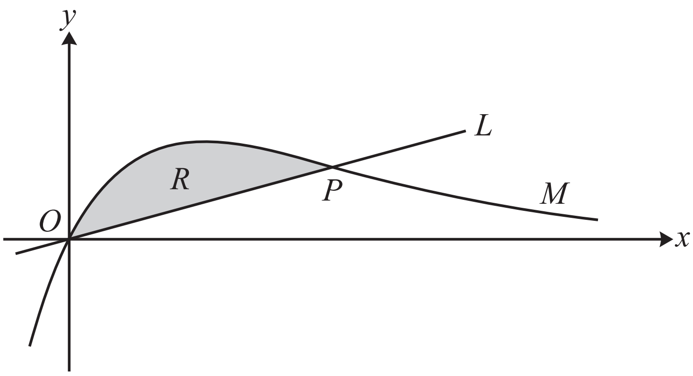
It is given that C passes through the point (4, 3) and that $x > 3$ and $y > 2$ at all points on C.

(b) Determine the equation of C giving your answer in the form $y = \mathrm{f}(x)$. **[4]**

The curve C may be obtained by a transformation of part of the curve $y = x^3$.

(c) Describe fully this transformation. **[2]**

8



The diagram shows the curve M with equation $y = xe^{-2x}$.

(a) Show that M has a point of inflection at the point P where $x = 1$. **[5]**

The line L passes through the origin O and the point P. The shaded region R is enclosed by the curve M and the line L.

(b) Show that the area of R is given by

$$\tfrac{1}{4}(a + be^{-2}),$$

where a and b are integers to be determined. **[6]**

Section B: Mechanics

Answer **all** the questions.

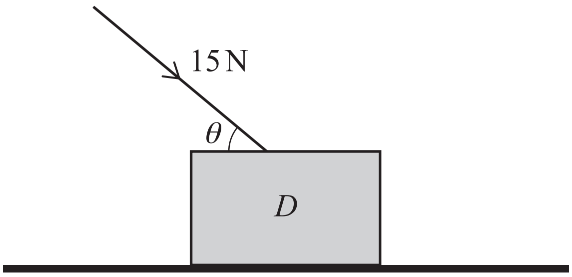
9 There are three checkpoints, A, B and C, in that order, on a straight horizontal road. A car travels along the road, in the direction from A to C, with constant acceleration. The car takes 20 s to travel from B to C. The speed of the car at B is $14\,\text{m s}^{-1}$ and the speed of the car at C is $18\,\text{m s}^{-1}$.

(a) Find the acceleration of the car. **[1]**

It is given that the distance between A and B is 330 m.

(b) Determine the speed of the car at A. **[2]**

10



A block D of weight 50 N lies at rest in equilibrium on a fixed rough horizontal surface. A force of magnitude 15 N is applied to D at an angle θ to the horizontal (see diagram).

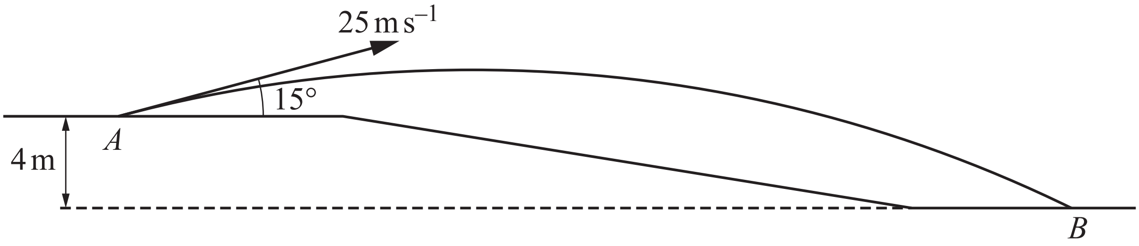
(a) Complete the diagram in the Printed Answer Booklet showing all the forces acting on D. **[1]**

It is given that D remains at rest and the coefficient of friction between D and the surface is 0.2.

(b) Show that

$$15\cos\theta - 3\sin\theta \leqslant 10.$$ **[5]**

11



A golfer hits a ball from a point A with a speed of $25\,\mathrm{m\,s^{-1}}$ at an angle of 15° above the horizontal. While the ball is in the air, it is modelled as a particle moving under the influence of gravity. Take the acceleration due to gravity to be $10\,\mathrm{m\,s^{-2}}$.

The ball first lands at a point B which is 4 m below the level of A (see diagram).

(a) Determine the time taken for the ball to travel from A to B. **[3]**

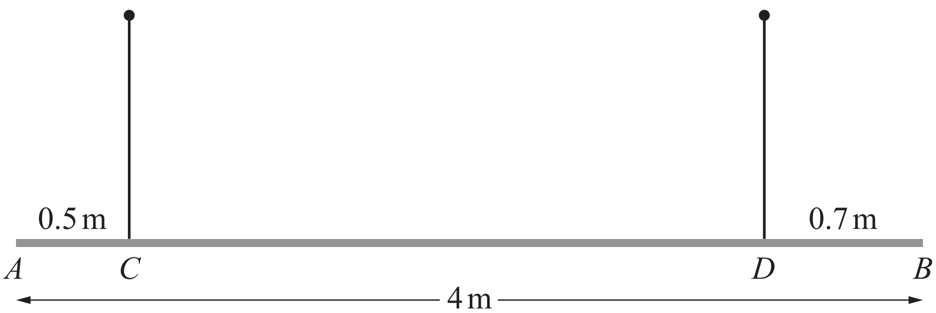
(b) Determine the horizontal distance of B from A. **[2]**

(c) Determine the direction of motion of the ball 1.5 seconds after the golfer hits the ball. **[4]**

The horizontal distance from A to B is found to be greater than the answer to part **(b)**.

(d) State one factor that could account for this difference. **[1]**

12



A beam, AB, has length 4 m and mass 20 kg. The beam is suspended horizontally by two vertical ropes. One rope is attached to the beam at C, where $AC = 0.5$ m. The other rope is attached to the beam at D, where $DB = 0.7$ m (see diagram).

The beam is modelled as a non-uniform rod and the ropes as light inextensible strings.

It is given that the tension in the rope at C is three times the tension in the rope at D.

(a) Determine the distance of the centre of mass of the beam from A. **[5]**

A particle of mass m kg is now placed on the beam at a point where the magnitude of the moment of the particle's weight about C is $3.5mg$ N m. The beam remains horizontal and in equilibrium.

(b) Determine the largest possible value of m. **[2]**

13 In this question the unit vectors **i** and **j** are in the directions east and north respectively.

At time t seconds, where $t \geqslant 0$, a particle P of mass $2\,\text{kg}$ is moving on a smooth horizontal surface under the action of a constant horizontal force $(-8\mathbf{i} - 54\mathbf{j})\,\text{N}$ and a variable horizontal force $\left(4t\mathbf{i} + 6(2t-1)^2\mathbf{j}\right)\text{N}$.

(a) Determine the value of t when the forces acting on P are in equilibrium. **[2]**

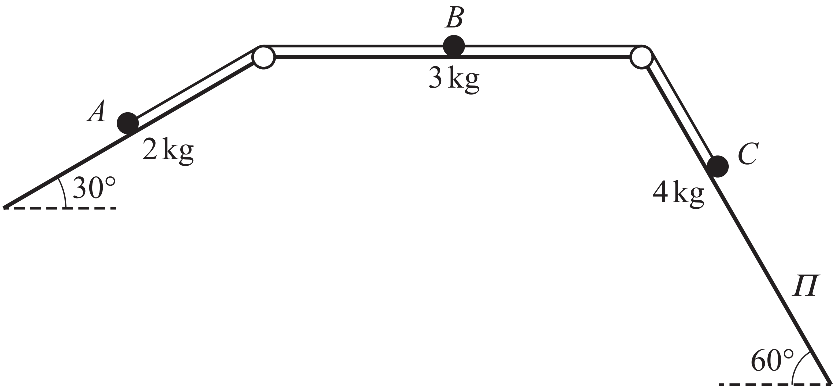
It is given that P is at rest when $t = 0$.

(b) Determine the speed of P at the instant when P is moving due north. **[6]**

(c) Determine the distance between the positions of P when $t = 0$ and $t = 3$. **[5]**

Turn over for question 14

14



One end of a light inextensible string is attached to a particle A of mass 2 kg. The other end of the string is attached to a second particle B of mass 3 kg. Particle A is in contact with a smooth plane inclined at 30° to the horizontal and particle B is in contact with a rough horizontal plane.

A second light inextensible string is attached to B. The other end of this second string is attached to a third particle C of mass 4 kg. Particle C is in contact with a smooth plane Π inclined at an angle of 60° to the horizontal.

Both strings are taut and pass over small smooth pulleys that are at the tops of the inclined planes. The parts of the strings from A to the pulley, and from C to the pulley, are parallel to lines of greatest slope of the corresponding planes (see diagram).

The coefficient of friction between B and the horizontal plane is μ. The system is released from rest and in the subsequent motion C moves down Π with acceleration $a \text{ m s}^{-2}$.

(a) By considering an equation involving μ, a and g show that $a < \frac{1}{9}g(2\sqrt{3}-1)$. **[7]**

(b) Given that $a = \frac{1}{9}g$, determine the magnitude of the contact force between B and the horizontal plane. Give your answer correct to **3** significant figures. **[4]**

END OF QUESTION PAPER

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