Do not write outside the box

Section A

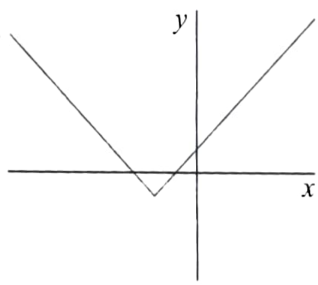
Answer **all** questions in the spaces provided.

1 Identify the graph of $y = 1 - |x + 2|$ from the options below.

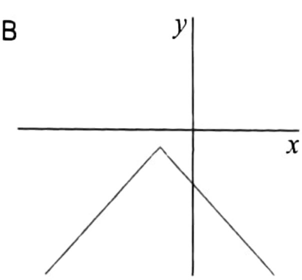
Tick (✓) **one** box.

[1 mark]

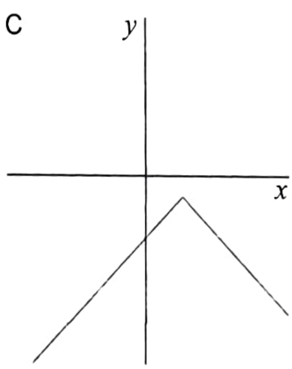
A



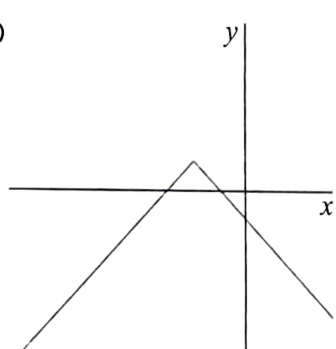
B



C



D



✓

2 Simplify $\sqrt{a^{\frac{2}{3}} \times a^{\frac{2}{5}}}$

Circle your answer.

[1 mark]

$a^{\frac{2}{15}}$ $\quad$ $a^{\frac{4}{15}}$ $\quad$ $a^{\frac{8}{15}}$ $\quad$ $a^{\frac{16}{15}}$

3 Each of these functions has domain $x \in \mathbb{R}$

Which function does **not** have an inverse?

Circle your answer.

[1 mark]

$\mathrm{f}(x) = x^3$ $\quad$ $\mathrm{f}(x) = 2x + 1$ $\quad$ $\mathrm{f}(x) = x^2$ $\quad$ $\mathrm{f}(x) = \mathrm{e}^x$

Turn over for the next question

Do not write outside the box

4 $x^2 + bx + c$ and $x^2 + dx + e$ have a common factor $(x + 2)$

Show that $2(d - b) = e - c$

Fully justify your answer.

[4 marks]

$(-2)^2 - 2b + c = 0$ by Factor Theorem

$4 - 2b + c = 0$

$(-2)^2 - 2d + e = 0$ by Factor Theorem.

$4 - 2d + e = 0$

$\Rightarrow \quad e - 2d = c - 2b$

$\Rightarrow \quad 2(d - b) = e - c$.

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5 Solve the differential equation

$$\frac{dt}{dx} = \frac{\ln x}{x^2 t} \qquad \text{for } x > 0$$

given $x = 1$ when $t = 2$

Write your answer in the form $t^2 = f(x)$

[7 marks]

$\int t\,dt = \int \frac{\ln x}{x^2}\,dx.$ $\quad u = \ln x \quad v = -x^{-1}.$

$u' = 1/x \quad v' = x^{-2}$

$\frac{t^2}{2} + c = \int \frac{\ln x}{x^2}\,dx.$

$= \frac{-\ln x}{x} + \int \frac{1}{x^2}\,dx$

$= \frac{-\ln x}{x} - \frac{1}{x} = \frac{-(\ln x + 1)}{x}$

$t = 2, \; x = 1:$

$\frac{-(\ln 1 + 1)}{1} = 2 + c$

$\Rightarrow c = -3.$

$\Rightarrow t^2 = 6 - 2\frac{\ln x + 1}{x}$

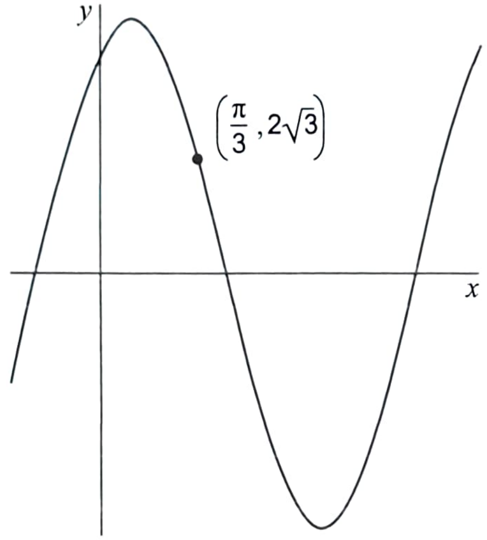
Turn over for the next question

6 A curve has equation

$$y = a\sin x + b\cos x$$

where a and b are constants.

The maximum value of y is 4 and the curve passes through the point $\left(\frac{\pi}{3}, 2\sqrt{3}\right)$ as shown in the diagram.



Find the exact values of a and b.

[6 marks]

$y = a\sin x + b\cos x = R\sin(x+\alpha)$.

$R = 4$ (by maximum value statement).

$4\sin\left(\frac{\pi}{3}+\alpha\right) = 2\sqrt{3}$.

$\Rightarrow \frac{\pi}{3}+\alpha = \frac{\pi}{3}, \frac{2\pi}{3}$.

$\Rightarrow \alpha = \frac{\pi}{3}$. This is not the correct solution (by the graph).

$4\sin\left(x+\frac{\pi}{3}\right)$

$= 4\cos\frac{\pi}{3}\sin x + 4\sin\frac{\pi}{3}\cos x$.

$\Rightarrow a = 4\cos\frac{\pi}{3} = 2$,

$b = 4\sin\frac{\pi}{3} = 2\sqrt{3}$.

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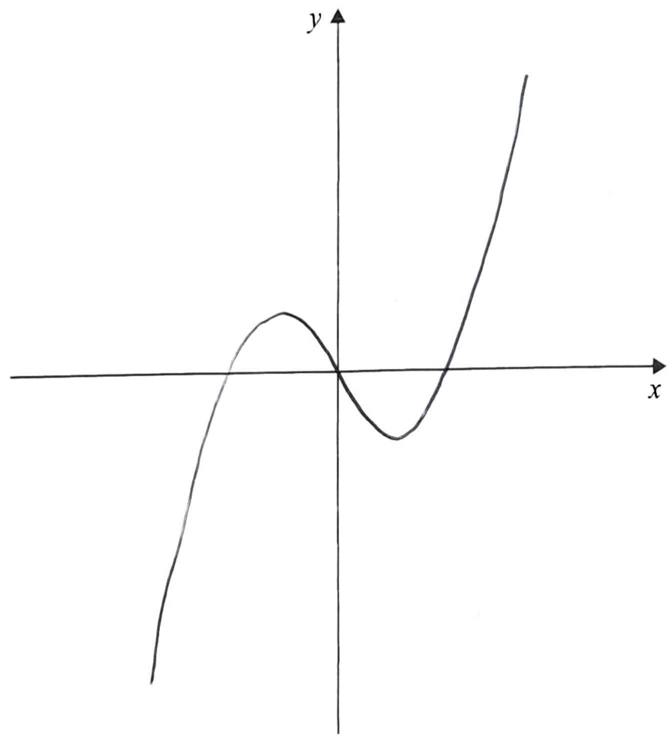
7 (a) Sketch the graph of any cubic function that has **both** three distinct real roots **and** a positive coefficient of x^3

[2 marks]



7 (b) The function $f(x)$ is defined by

$$f(x) = x^3 + 3px^2 + q$$

where p and q are constants and $p > 0$

7 (b) (i) Show that there is a turning point where the curve crosses the y-axis.

[3 marks]

$f(x) = x^3 + 3px^2 + q$

$\Rightarrow f'(x) = 3x^2 + 6px$

$3x^2 + 6px = 0$

$3x(x + 2p) = 0$

$\Rightarrow x = 0, -2p.$

This is a turning point on the y-axis.

7 (b) (ii) The equation $f(x) = 0$ has three distinct real roots.

By considering the positions of the turning points find, in terms of p, the range of possible values of q.

[5 marks]

Since $p > 0$, $x = -2p$ must be the maximum (using the sketch in part (a)), and $x = 0$ the minimum.

$f(0) = q$, $f(-2p) = -8p^3 + 12p^3 + q$
$= 4p^3 + q > q$.

$q < 0$ since there are real roots.
Since $p > 0 > q$, we have $p > q$
$\Rightarrow -4p^3 < q$.

$\Rightarrow -4p^3 < q < 0$.

Turn over for the next question

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8 Theresa bought a house on 2 January 1970 for £8000.

The house was valued by a local estate agent on the same date every 10 years up to 2010.

The valuations are shown in the following table.

Year	1970	1980	1990	2000	2010
Valuation price	£8 000	£19 000	£36 000	£82 000	£205 000

The valuation price of the house can be modelled by the equation

$$V = pq^t$$

where V pounds is the valuation price t years after 2 January 1970 and p and q are constants.

8 (a) Show that $V = pq^t$ can be written as $\log_{10} V = \log_{10} p + t \log_{10} q$

[2 marks]

$\log_{10} V = \log_{10}(pq^t) = \log_{10} p + \log_{10}(q^t)$

$= \log_{10} p + t \log_{10} q$.

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Using the given line of best fit, find estimates for the values of p and q.

Give your answers correct to three significant figures.

[4 marks]

$\log_{10} p = 3.9. \Rightarrow \quad p = 10^{3.9} = 7940.$

$\log_{10} q = \dfrac{5.28 - 3.9}{40 - 0} \Rightarrow q = 10^{0.0345} = 1.08.$

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8 (c) Determine the year in which Theresa's house will first be worth half a million pounds.

[3 marks]

$V = 500\,000$ [illegible]

$500\,000 = 7940 \times 1.08^t$

$\Rightarrow 1.08^t = \frac{500000}{7940}$

$t = \frac{\log \frac{500000}{7940}}{\log 1.08} = 53.83$

It will be worth this much in 2023.

8 (d) Explain whether your answer to part **(c)** is likely to be reliable.

[2 marks]

We cannot extrapolate the model beyond 2010, so the figure may not be accurate.

Turn over for the next question

Do not write outside the box

9 (a) Show that the first two terms of the binomial expansion of $\sqrt{4-2x^2}$ are

$$2-\frac{x^2}{2}$$

[2 marks]

$\sqrt{4-2x^2} = \sqrt{4}\sqrt{1-\frac{1}{2}x^2} = 2\sqrt{1-\frac{1}{2}x^2}$

$= 2\left(1-\frac{1}{2}x^2\right)^{\frac{1}{2}}$

$= 2\left(1+\frac{1}{2}\left(-\frac{1}{2}x^2\right)+\ldots\right)$

$= 2 - \frac{x^2}{2}$.

9 (b) State the range of values of x for which the expansion found in part **(a)** is valid.

[2 marks]

$\left|-\frac{x^2}{2}\right| < 1$

$\Rightarrow |x^2| < 2$

$\Rightarrow |x| < \sqrt{2}$.

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9 (c) Hence, find an approximation for

$$\int_0^{0.4} \sqrt{\cos x}\, dx$$

giving your answer to five decimal places.

Fully justify your answer.

[4 marks]

0.4 is small, so we can use approximations.

$$\cos x \approx 1 - \frac{x^2}{2}$$

$$\int_0^{0.4} \sqrt{\cos x}\, dx \approx \int_0^{0.4} \sqrt{1 - \frac{x^2}{2}}\, dx$$

$$\approx \frac{1}{2}\int_0^{0.4} 2 - \frac{x^2}{2}\, dx = \int_0^{0.4} 1 - \frac{x^2}{4}\, dx$$

$$= \left[x - \frac{x^3}{12}\right]_0^{0.4} = \left(0.4 - \frac{2}{375}\right) = 0.39467.$$

9 (d) A student decides to use this method to find an approximation for

$$\int_0^{1.4} \sqrt{\cos x}\, dx$$

Explain why this may not be a suitable method.

[1 mark]

1.4 is not a small angle, so SA. approximations are not suitable.

Turn over for Section B

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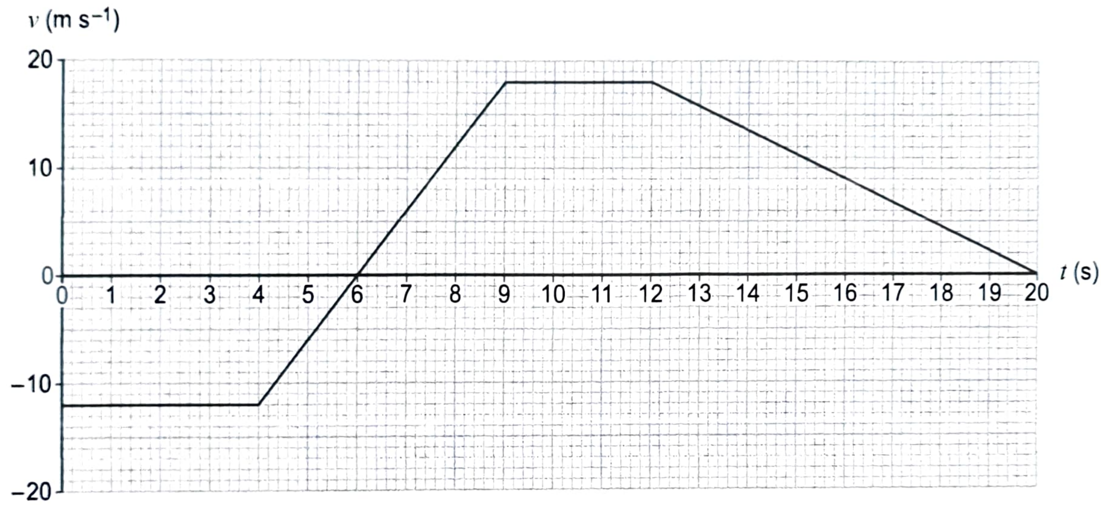
Section B

Answer **all** questions in the spaces provided.

10 The diagram below shows a velocity-time graph for a particle moving with velocity $v\,\mathrm{m\,s^{-1}}$ at time t seconds.



Which statement is correct?

Tick (✓) **one** box.

[1 mark]

- [ ] The particle was stationary for $9 \le t \le 12$
- [x] The particle was decelerating for $12 \le t \le 20$
- [ ] The particle had a displacement of zero when $t = 6$
- [ ] The particle's speed when $t = 4$ was $-12\,\mathrm{m\,s^{-1}}$

11 A wooden crate rests on a rough horizontal surface.

The coefficient of friction between the crate and the surface is 0.6

A forward force acts on the crate, parallel to the surface.

When this force is 600 N, the crate is on the point of moving.

Find the weight of the crate.

Circle your answer.

[1 mark]

1000 N   100 kg   360 N   36 kg

12 A particle, under the action of two constant forces, is moving across a perfectly smooth horizontal surface at a constant speed of $10\,\text{m s}^{-1}$

The first force acting on the particle is $(400\mathbf{i} + 180\mathbf{j})$ N.

The second force acting on the particle is $(p\mathbf{i} - 180\mathbf{j})$ N.

Find the value of p.

Circle your answer.

[1 mark]

−400   −390   390   400

Turn over for the next question

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13 In a school experiment, a particle, of mass m kilograms, is released from rest at a point h metres above the ground.

At the instant it reaches the ground, the particle has velocity $v\,\mathrm{m\,s^{-1}}$

13 (a) Show that

$$v = \sqrt{2gh}$$

[2 marks]

$s = h,\quad u = 0,\quad v = v,\quad a = g.$

$v^2 = u^2 + 2as$

$\Rightarrow \quad v^2 = 2gh$

$\Rightarrow \quad v = \sqrt{2gh}$

13 (b) A student correctly used $h = 18$ and measured v as 20

The student's teacher claims that the machine measuring the velocity must have been faulty.

Determine if the teacher's claim is correct.

Fully justify your answer.

[3 marks]

$v = \sqrt{2 \times 9.8 \times 18} = 18.78.$

$18.78 < 20$

Yes, their ~~that~~ claim is correct.

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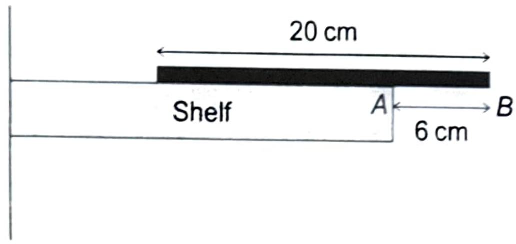
14 A metal rod, of mass m kilograms and length 20 cm, lies at rest on a horizontal shelf.

The end of the rod, B, extends 6 cm beyond the edge of the shelf, A, as shown in the diagram below.



14 (a) The rod is in equilibrium when an object of mass 0.28 kilograms hangs from the midpoint of AB.

Show that $m = 0.21$

[3 marks]

About A:

CW: $0.3 \times 0.28g$

ACW: $0.4 \times$ [redacted] mg

$\Rightarrow m = \frac{0.3}{0.4} 0.28 = 0.21$ kg.

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14 (b) The object of mass 0.28 kilograms is removed.

A number, n, of identical objects, each of mass 0.048 kg, are hung from the rod all at a distance of 1 cm from B.

Find the maximum value of n such that the rod remains horizontal.

[4 marks]

ACW: $0.21 \times 4 \times g$.

CW: $5 \times 0.048n \times g$.

$\Rightarrow \quad 0.048n = \frac{0.21 \times \cancel{5}4}{5}$

$\Rightarrow \quad n = 3.5$.

The maximum value is $n = 3$.

14 (c) State one assumption you have made about the rod.

[1 mark]

The rod is uniformly weighted.

Turn over for the next question

Do not write outside the box

15 Four buoys on the surface of a large, calm lake are located at A, B, C and D with position vectors given by

$$\overrightarrow{OA} = \begin{bmatrix} 410 \\ 710 \end{bmatrix}, \overrightarrow{OB} = \begin{bmatrix} -210 \\ 530 \end{bmatrix}, \overrightarrow{OC} = \begin{bmatrix} -340 \\ -310 \end{bmatrix} \text{ and } \overrightarrow{OD} = \begin{bmatrix} 590 \\ -40 \end{bmatrix}$$

All values are in metres.

15 (a) Prove that the quadrilateral $ABCD$ is a trapezium but **not** a parallelogram.

[5 marks]

$\overrightarrow{AB} = \begin{bmatrix} -620 \\ -180 \end{bmatrix}$, $\overrightarrow{CD} = \begin{bmatrix} 930 \\ 270 \end{bmatrix}$

$\Rightarrow \overrightarrow{CD} = -1.5\overrightarrow{AB}$

So, CD and AB are parallel, but not of equal length.

So, ABCD is a trapezium, but not a parallelogram.

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15 (b) A speed boat travels directly from B to C at a constant speed in 50 seconds.

Find the speed of the boat between B and C.

[4 marks]

$\overrightarrow{BC} = \begin{bmatrix} -130 \\ -840 \end{bmatrix}.$

$\underline{v} = \frac{1}{50}\begin{bmatrix} -130 \\ -840 \end{bmatrix} = \begin{bmatrix} -2.6 \\ -16.8 \end{bmatrix}$

$|\underline{v}| = \sqrt{2.6^2 + 16.8^2} = \sqrt{289} = 17 \text{ ms}^{-1}.$

Turn over for the next question

Do not write outside the box

16 An elite athlete runs in a straight line to complete a 100-metre race.

During the race, the athlete's velocity, $v\,\mathrm{m\,s^{-1}}$, may be modelled by

$$v = 11.71 - 11.68e^{-0.9t} - 0.03e^{0.3t}$$

where t is the time, in seconds, after the starting pistol is fired.

16 (a) Find the maximum value of v, giving your answer to one decimal place.

Fully justify your answer.

[8 marks]

$V = 11.71 - 11.68e^{-0.9t} - 0.03e^{0.3t}$

$\frac{dv}{dt} = 10.512e^{-0.9t} - 0.009e^{0.3t} = 0.$

$10.512e^{-0.9t} = 0.009e^{0.3t}$

$1168 = e^{1.2t}.$

$1.2t = \ln 1168$

$\Rightarrow t = 5.886\,s.$

$V_{t=5.886} = 11.476 \rightarrow 11.5\,ms^{-1}.$

This is ~~the~~ ~~for~~ only value of t such that $\frac{dv}{dt} = 0$, so it must be the maximum.

16 (b) Find an expression for the distance run in terms of t.

[6 marks]

$s = \int v \, dt$

$= \int 11.71 - 11.68e^{-0.9t} - 0.03e^{0.3t} \, dt$

$= 11.71t + 12.978e^{-0.9t} - 0.1e^{0.3t} + C$

$S = 0$ when $t = 0$, so,

$0 = 12.978 - 0.1 + c$

$\Rightarrow c = -12.878.$

$\Rightarrow S = 11.71t + 12.978e^{-0.9t} - 0.1e^{0.3t} - 12.878.$

Question 16 continues on the next page

Turn over ▶

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16 (c) The athlete's actual time for this race is 9.8 seconds.

Comment on the accuracy of the model.

[2 marks]

$t = 9.8 \Rightarrow$

$S = 99.99m.$

This is very close to 100m, so the model is very accurate.

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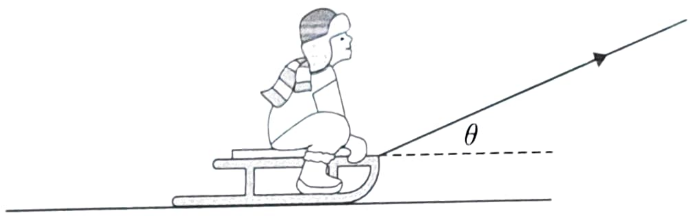
17 Lizzie is sat securely on a wooden sledge.

The combined mass of Lizzie and the sledge is M kilograms.

The sledge is being pulled forward in a straight line along a horizontal surface by means of a light inextensible rope, which is attached to the front of the sledge.

This rope stays inclined at an acute angle θ above the horizontal and remains taut as the sledge moves forward.



The sledge remains in contact with the surface throughout.

The coefficient of friction between the sledge and the surface is μ and there are no other resistance forces.

Lizzie and the sledge move forward with constant acceleration, $a\,\mathrm{m\,s^{-2}}$

The tension in the rope is a constant T Newtons.

17 (a) Show that

$$T = \frac{M(a+\mu g)}{\cos\theta + \mu\sin\theta}$$

[7 marks]

Vertically: $R + T\sin\theta = Mg$

Horizontally: $T\cos\theta - \text{Friction} = Ma.$

Friction $= \mu R.$

$\Rightarrow T\cos\theta - \mu R = Ma.$

$\Rightarrow T\cos\theta - \mu(Mg - T\sin\theta) = Ma$

$\Rightarrow T(\cos\theta + \mu\sin\theta) - \mu Mg = Ma$

$\Rightarrow T(\cos\theta + \mu\sin\theta) = M(a + \mu g)$

$\Rightarrow T = \dfrac{M(a+\mu g)}{\cos\theta + \mu\sin\theta}$

17 (b) It is known that when $M = 30$, $\theta = 30°$, and $T = 40$, the sledge remains at rest.

Lizzie uses these values with the relationship formed in part **(a)** to find the value for μ

Explain why her value for μ may be incorrect.

[2 marks]

$$40 = \frac{30 \times 9.8\mu}{\frac{\sqrt{3}}{2} + \frac{1}{2}\mu} \Rightarrow 20\sqrt{3} + 20\mu = 294\mu$$

$$\Rightarrow \ldots$$

The sledge is at rest, so friction may not be at its limiting value.

END OF QUESTIONS