Answer ALL questions. Write your answers in the spaces provided.

1.
$$f(x) = 3x^3 + 2ax^2 - 4x + 5a$$

Given that $(x + 3)$ is a factor of $f(x)$, find the value of the constant a.

(3)

$f(-3) = 0$.

$3(-3)^3 + 2a(-3)^2 - 4(-3) + 5a = 0$

$\Rightarrow -81 + 18a + 12 + 5a = 0$

$\Rightarrow 23a - 69 = 0$

$\Rightarrow a = 3$.

2.

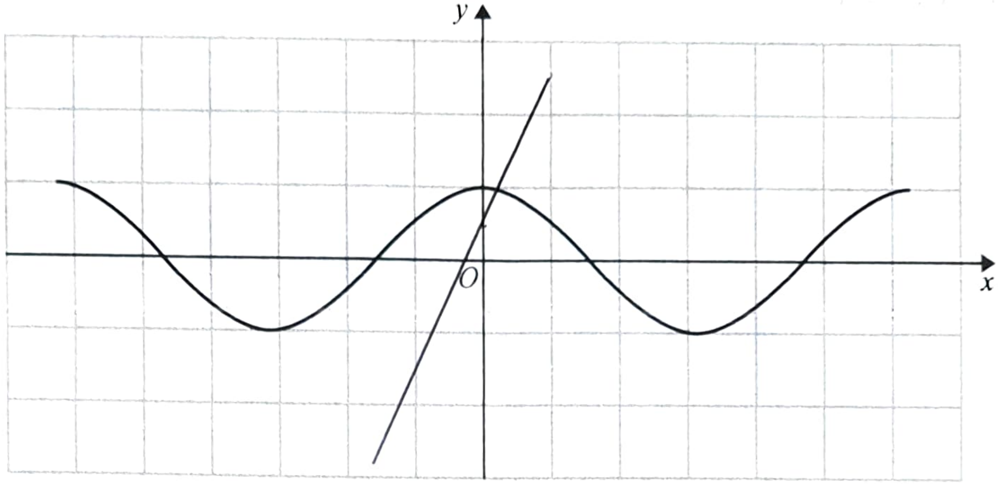

Figure 1

Figure 1 shows a plot of part of the curve with equation $y = \cos x$ where x is measured in radians. Diagram 1, on the opposite page, is a copy of Figure 1.

(a) Use Diagram 1 to show why the equation

$$\cos x - 2x - \frac{1}{2} = 0$$

has only one real root, giving a reason for your answer.

(2)

Given that the root of the equation is α, and that α is small,

(b) use the small angle approximation for $\cos x$ to estimate the value of α to 3 decimal places.

(3)

a) There is only one intersection.

b) $\cos\alpha = 1 - \frac{1}{2}\alpha^2$

$\cos\alpha - 2\alpha - \frac{1}{2} = 0$

$\Rightarrow 1 - \frac{1}{2}\alpha^2 - 2\alpha - \frac{1}{2} = 0.$

$\Rightarrow \frac{1}{2}\alpha^2 + 2\alpha - \frac{1}{2} = 0$

$\Rightarrow \alpha^2 + 4\alpha - 1 = 0$

$\Rightarrow \alpha = \dfrac{-4 \pm \sqrt{16+4}}{2} = -2 \pm \sqrt{5}$

$= 0.236$

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3.
$$y = \frac{5x^2 + 10x}{(x+1)^2} \qquad x \neq -1$$

(a) Show that $\dfrac{dy}{dx} = \dfrac{A}{(x+1)^n}$ where A and n are constants to be found.

(4)

(b) Hence deduce the range of values for x for which $\dfrac{dy}{dx} < 0$

(1)

a) $f = 5x^2 + 10x \qquad g = (x+1)^2$

$f' = 10x + 10 \qquad g' = 2(x+1)$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{10(x+1)^3 - 10x(x+2)(x+1)}{(x+1)^4}$.

$= \dfrac{10[(x+1)^2 - x(x+2)]}{(x+1)^3}$

$= \dfrac{10(x^2+2x+1 - x^2 - 2x)}{(x+1)^3} = \dfrac{10}{(x+1)^3}$

b) $x < -1$.

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4. (a) Find the first three terms, in ascending powers of x, of the binomial expansion of

$$\frac{1}{\sqrt{4-x}}$$

giving each coefficient in its simplest form.

(4)

The expansion can be used to find an approximation to $\sqrt{2}$
Possible values of x that could be substituted into this expansion are:

- $x = -14$ because $\dfrac{1}{\sqrt{4-x}} = \dfrac{1}{\sqrt{18}} = \dfrac{\sqrt{2}}{6}$
- $x = 2$ because $\dfrac{1}{\sqrt{4-x}} = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$
- $x = -\dfrac{1}{2}$ because $\dfrac{1}{\sqrt{4-x}} = \dfrac{1}{\sqrt{\dfrac{9}{2}}} = \dfrac{\sqrt{2}}{3}$

(b) Without evaluating your expansion,

(i) state, giving a reason, which of the three values of x should not be used

(1)

(ii) state, giving a reason, which of the three values of x would lead to the most accurate approximation to $\sqrt{2}$

(1)

a) $(4-x)^{-\frac{1}{2}} = 4^{-\frac{1}{2}}\left(1-\frac{x}{4}\right)^{-\frac{1}{2}}$

$= \frac{1}{2}\left(1 + \frac{x}{8} + \frac{3}{128}x^2\right)$

$= \frac{1}{2} + \frac{1}{16}x + \frac{3}{256}x^2$.

b) i) $x = -14$. It is outside of the range $|x| < 4$.

ii) $x = -\frac{1}{2}$, as it the closest to 0.

5. $$f(x) = 2x^2 + 4x + 9 \qquad x \in \mathbb{R}$$

(a) Write f(x) in the form $a(x + b)^2 + c$, where a, b and c are integers to be found.
(3)

(b) Sketch the curve with equation $y = f(x)$ showing any points of intersection with the coordinate axes and the coordinates of any turning point.
(3)

(c) (i) Describe fully the transformation that maps the curve with equation $y = f(x)$ onto the curve with equation $y = g(x)$ where

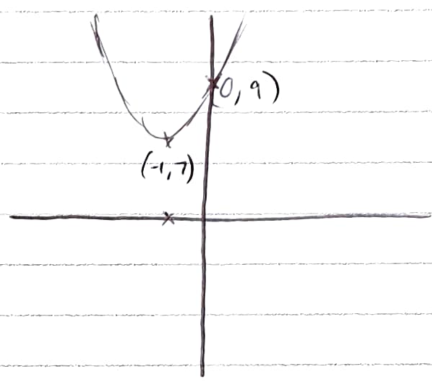
$$g(x) = 2(x - 2)^2 + 4x - 3 \qquad x \in \mathbb{R}$$

(ii) Find the range of the function

$$h(x) = \frac{21}{2x^2 + 4x + 9} \qquad x \in \mathbb{R}$$
(4)

a) $2(x+1)^2 - 2 + 9 = 2(x+1)^2 + 7.$

b)



c) i) $2(x-2)^2 + 4x - 3 = 2(x-2)^2 + 4(x-2) + 5$

Translation in $\begin{bmatrix} 2 \\ -4 \end{bmatrix}$

ii) $\frac{21}{2(x+1)^2+7}$. Minimum value tends towards 0.

Maximum value given when $x = -1$, i.e. $h(x) = \frac{21}{7} = 3$.

$0 < h(x) \leq 3.$

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6. (a) Solve, for $-180° \leqslant \theta \leqslant 180°$, the equation

$$5\sin 2\theta = 9\tan\theta$$

giving your answers, where necessary, to one decimal place.

[*Solutions based entirely on graphical or numerical methods are not acceptable.*]

(6)

(b) Deduce the smallest positive solution to the equation

$$5\sin(2x - 50°) = 9\tan(x - 25°)$$

(2)

a) $5\sin 2\theta = 10\sin\theta\cos\theta.$

$10\sin\theta\cos\theta = 9\frac{\sin\theta}{\cos\theta}.$

$\Rightarrow 10\cos^2\theta = 9.$

$\Rightarrow \cos\theta = \pm\frac{3}{\sqrt{10}}$

$\Rightarrow \theta = \cos^{-1}\left(\pm\frac{3}{\sqrt{10}}\right)$

$= \pm 18.4°, \pm 161.6°, 0°, \pm 180°.$

b) $x - 25° = -18.4°$

$\Rightarrow x = 6.6°.$

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7. In a simple model, the value, £V, of a car depends on its age, t, in years.

The following information is available for car A

- its value when new is £20 000
- its value after one year is £16 000

(a) Use an exponential model to form, for car A, a possible equation linking V with t.

(4)

The value of car A is monitored over a 10-year period.
Its value after 10 years is £2 000

(b) Evaluate the reliability of your model in light of this information.

(2)

The following information is available for car B

- it has the same value, when new, as car A
- its value depreciates more slowly than that of car A

(c) Explain how you would adapt the equation found in (a) so that it could be used to model the value of car B.

(1)

a) $V = ae^{kt}$

$t=0 \Rightarrow V = 20000$
$t=1 \Rightarrow V = 16000$

$a = 20000$.

$16000 = 20000\,e^{k}$

$\Rightarrow k = \ln \frac{16000}{20000} = -0.223$.

$\Rightarrow V = 20000e^{-0.223t}$

b) $t=10 \Rightarrow V = 2147.50 \approx 2000$

The model is fairly reliable.

Question 7 continued

c) Make the value of k less negative.

8.

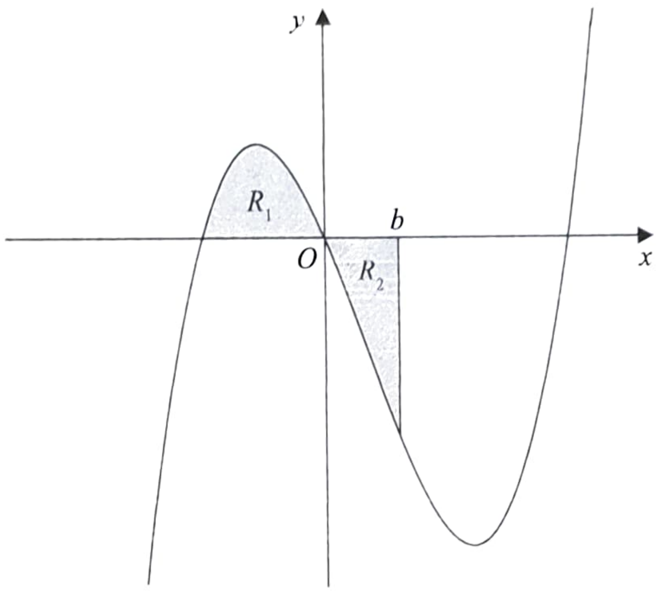

Figure 2

Figure 2 shows a sketch of part of the curve with equation $y = x(x + 2)(x - 4)$.

The region R_1 shown shaded in Figure 2 is bounded by the curve and the negative x-axis.

(a) Show that the exact area of R_1 is $\frac{20}{3}$

(4)

The region R_2 also shown shaded in Figure 2 is bounded by the curve, the positive x-axis and the line with equation $x = b$, where b is a positive constant and $0 < b < 4$

Given that the area of R_1 is equal to the area of R_2

(b) verify that b satisfies the equation

$$(b + 2)^2 (3b^2 - 20b + 20) = 0$$

(4)

The roots of the equation $3b^2 - 20b + 20 = 0$ are 1.225 and 5.442 to 3 decimal places. The value of b is therefore 1.225 to 3 decimal places.

(c) Explain, with the aid of a diagram, the significance of the root 5.442

(2)

a) $y = x(x+2)(x-4) = x(x^2-2x-8) = x^3-2x^2-8x$

$\int_{-2}^{0} x^3 - 2x^2 - 8x \, dx = \left[\frac{1}{4}x^4 - \frac{2}{3}x^3 - 4x^2\right]_{-2}^{0}$

$= 0 - \left(4 + \frac{16}{3} - 16\right) = \frac{20}{3}$

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Question 8 continued

b) $\left[\frac{1}{4}x^4 - \frac{2}{3}x^3 - 4x^2\right]_0^b = \frac{-20}{3}$

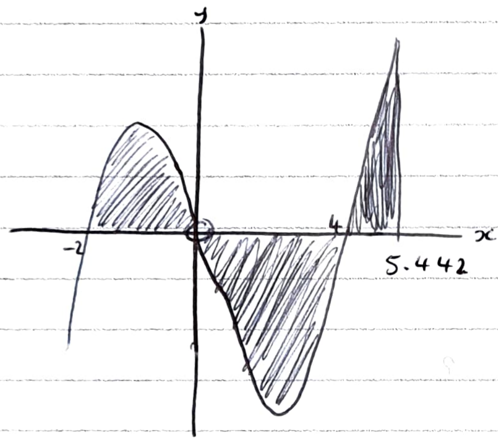
$\Rightarrow \frac{1}{4}b^4 - \frac{2}{3}b^3 - 4b^2 = \frac{-20}{3}$

$\Rightarrow 3b^4 - 8b^3 - 48b^2 + 80 = 0.$

$\Rightarrow (b+2)(3b^3 - 14b^2 - 20b + 40) = 0$

$\Rightarrow (b+2)^2(3b^2 - 20b + 20) = 0.$

c) Between $x=-2$ and $x=5.442$, the integral is 0. i.e. the area above the x-axis is equal to the area below.



Turn over

9. Given that $a > b > 0$ and that a and b satisfy the equation

$$\log a - \log b = \log(a - b)$$

(a) show that

$$a = \frac{b^2}{b-1}$$

(3)

(b) Write down the full restriction on the value of b, explaining the reason for this restriction.

(2)

a) $\log a - \log b = \log \frac{a}{b}$.

$\frac{a}{b} = a - b$.

$\Rightarrow a = ab - b^2$

$\Rightarrow b^2 = ab - a$

$\Rightarrow b^2 = a(b-1)$

$\Rightarrow a = \frac{b^2}{b-1}$

b) a must be greater than 0, so we require $b > 1$.

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10. (i) Prove that for all $n \in \mathbb{N}$, $n^2 + 2$ is not divisible by 4

(4)

(ii) "Given $x \in \mathbb{R}$, the value of $|3x - 28|$ is greater than or equal to the value of $(x - 9)$."
State, giving a reason, if the above statement is always true, sometimes true or never true.

(2)

i) n odd: $n = 2k+1$.

$$(2k+1)^2 + 2 = 4k^2 + 4k + 1 + 2$$
$$= 4(k^2 + k) + 3$$

k^2+k is an integer, so,

$$\frac{4(k^2+k)+3}{4} = \underbrace{k^2+k}_{\text{integer}} + \underbrace{3/4}_{\text{non-integer}}.$$

This is not divisible by 4.

n even: $n = 2k$.

$$(2k)^2 + 2 = 4k^2 + 2$$

k^2 is an integer, so,

$$\frac{4k^2+2}{4} = \underbrace{k^2}_{\text{integer}} + \underbrace{1/2}_{\text{non-integer}}.$$

This is not divisible by 4.

ii) The statement is sometimes true. Consider the point where $|3x-28| = 0$, $x = \frac{28}{3}$.

$|3x-28| = 0$, but $x - 9 = \frac{1}{3}$.
In this instance, $x - 9 > |3x-28|$.

Question 10 continued

However, $x=0$ gives $|3x-28|=28$ and $x-9=-9$.
In this instance, $|3x-28| > x-9$.

(Total for Question 10 is 6 marks)

11. A competitor is running a 20 kilometre race.

She runs each of the first 4 kilometres at a steady pace of 6 minutes per kilometre. After the first 4 kilometres, she begins to slow down.

In order to estimate her finishing time, the time that she will take to complete each subsequent kilometre is modelled to be 5% greater than the time that she took to complete the previous kilometre.

Using the model,

(a) show that her time to run the first 6 kilometres is estimated to be 36 minutes 55 seconds, **(2)**

(b) show that her estimated time, in minutes, to run the rth kilometre, for $5 \leqslant r \leqslant 20$, is

$$6 \times 1.05^{r-4}$$

(1)

(c) estimate the total time, in minutes and seconds, that she will take to complete the race. **(4)**

a) $(4\times6) + (6\times1.05) + (6\times1.05^2) = 36.915$ mins

$= 36$ mins, 55 seconds.

b) Let $r=5$: $6\times1.05^1 = 6.3$

Let $r=6$: $6\times1.05^2 = 6.615$

Let $r=7$: $6\times1.05^3 = 6.94575$.

So the r^{th} km takes $6\times1.05^{r-4}$ minutes.

c) Total time: $24 + \sum_{r=5}^{20} 6\times1.05^{r-4}$

$= 24 + \sum_{r=1}^{16} 6\times1.05^{r}$.

$= 24 + \dfrac{6.3(1.05^{16}-1)}{1.05-1}$

$= 24 + 149.04 = 173.04$ mins.

$= 173$ mins, 3 seconds.

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12. $$f(x) = 10e^{-0.25x} \sin x, \qquad x \geqslant 0$$

(a) Show that the x coordinates of the turning points of the curve with equation $y = f(x)$ satisfy the equation $\tan x = 4$

(4)

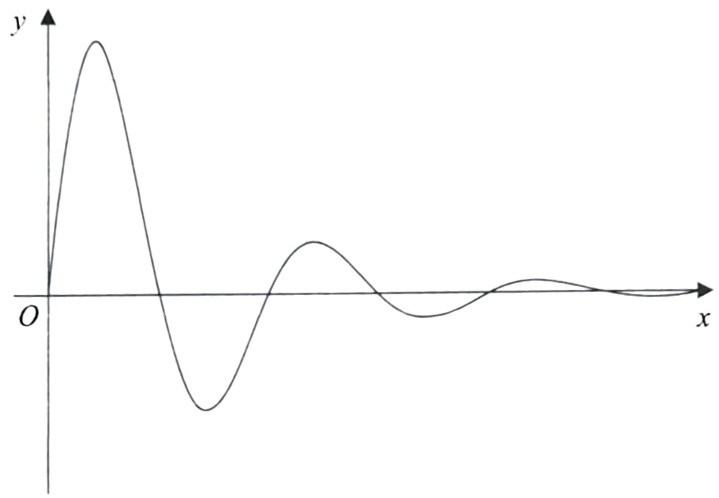

Figure 3

Figure 3 shows a sketch of part of the curve with equation $y = f(x)$.

(b) Sketch the graph of H against t where

$$H(t) = \left|10e^{-0.25t} \sin t\right| \qquad t \geqslant 0$$

showing the long-term behaviour of this curve.

(2)

The function $H(t)$ is used to model the height, in metres, of a ball above the ground t seconds after it has been kicked.

Using this model, find

(c) the maximum height of the ball above the ground between the first and second bounce.

(3)

(d) Explain why this model should not be used to predict the time of each bounce.

(1)

a) $f(x) = 10e^{-0.25x} \sin x$

$u = 10e^{-0.25x} \qquad v = \sin x$
$u' = -2.5e^{-0.25x} \qquad v' = \cos x$

$\Rightarrow f'(x) = 10e^{-0.25x}\cos x - 2.5e^{-0.25x}\sin x = 0$
$= e^{-0.25x}(10\cos x - 2.5\sin x) = 0$

$\Rightarrow \quad 10\cos x = 2.5\sin x$

$\Rightarrow \quad \tan x = 4.$

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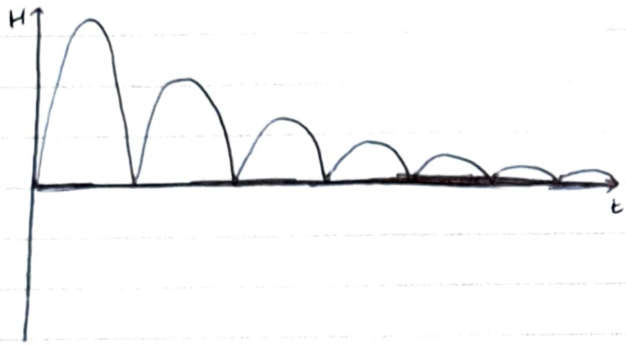
Question 12 continued

b)



c) $\tan x = 4 \Rightarrow x = 4.467.$

$$H(4.467) = \left|10e^{-0.25\times 4.467}\sin 4.467\right|$$
$$= |-3.175|$$
$$= 3.18\text{m.}$$

d) The time between bounces should decrease, as the height of the bounce does.

13. The curve C with equation

$$y = \frac{p - 3x}{(2x - q)(x + 3)} \qquad x \in \mathbb{R}, x \neq -3, x \neq 2$$

where p and q are constants, passes through the point $\left(3, \frac{1}{2}\right)$ and has two vertical asymptotes with equations $x = 2$ and $x = -3$

(a) (i) Explain why you can deduce that $q = 4$

(ii) Show that $p = 15$

(3)

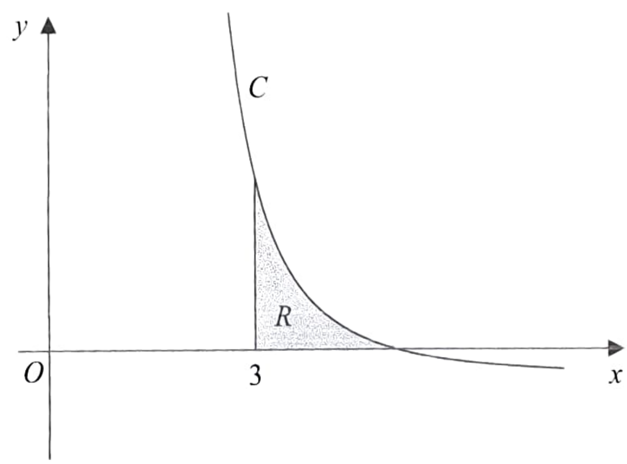


Figure 4

Figure 4 shows a sketch of part of the curve C. The region R, shown shaded in Figure 4, is bounded by the curve C, the x-axis and the line with equation $x = 3$

(b) Show that the exact value of the area of R is $a \ln 2 + b \ln 3$, where a and b are rational constants to be found.

(8)

a) i) The asymptote $x=2$ applies to $2x-q=0$.
$2(2) - q = 0 \Rightarrow q = 2\times 2 = 4$.

ii) $\frac{1}{2} = \frac{p-9}{2\times 6} \Rightarrow p-9 = 6 \Rightarrow p = 15$.

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Question 13 continued

b) R bounded between $x=3$, $x=5$.

$$\frac{15-3x}{(2x-4)(x+3)} = \frac{A}{2x-4} + \frac{B}{x+3}$$

$$\Rightarrow A(x+3) + B(2x-4) = 15-3x$$

$A + 2B = -3 \Rightarrow 2A + 4B = -6$
$3A - 4B = 15$

$$\Rightarrow 5A = 9 \Rightarrow A = 1.8,\ B = -2.4$$

$$\Rightarrow R = \int_3^5 \frac{1.8}{2x-4} - \frac{2.4}{x+3}\,dx$$

$$= \left[0.9\ln|2x-4| - 2.4\ln|x+3|\right]_3^5$$

$$= (0.9\ln 6 - 2.4\ln 8) - (0.9\ln 2 - 2.4\ln 6)$$

$$= (0.9\ln 6 + 2.4\ln 6) + (-7.2\ln 2 - 0.9\ln 2)$$

$$= 3.3\ln 6 - 8.1\ln 2$$

$$\boxed{\ln 6 = \ln 3 + \ln 2.}$$

$$= 3.3\ln 3 + 3.3\ln 2 - 8.1\ln 2$$
$$= 3.3\ln 3 - 4.8\ln 2.$$

14. The curve C, in the standard Cartesian plane, is defined by the equation

$$x = 4\sin 2y \qquad \frac{-\pi}{4} < y < \frac{\pi}{4}$$

The curve C passes through the origin O

(a) Find the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at the origin.

(2)

(b) (i) Use the small angle approximation for $\sin 2y$ to find an equation linking x and y for points close to the origin.

(ii) Explain the relationship between the answers to (a) and (b)(i).

(2)

(c) Show that, for all points (x, y) lying on C,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{a\sqrt{b - x^2}}$$

where a and b are constants to be found.

(3)

a) $\frac{dx}{dy} = 8\cos 2y$.

$\frac{dy}{dx} = \frac{1}{8\cos 2y}$

$y = 0 \Rightarrow \quad \frac{dy}{dx} = \frac{1}{8}$.

b) i) $\sin 2y \approx 2y \quad \Rightarrow \quad x \approx 8y$.

ii) Rearranging to $y = \frac{1}{8}x$ gives the gradient suggested in part a, $\frac{1}{8}$.

c) $\frac{dy}{dx} = \frac{1}{8\cos 2y}$

$\cos^2 2y + \sin^2 2y = 1$

[redacted]

$\Rightarrow \cos^2 2y + \frac{x^2}{16} = 1$.

$\Rightarrow \cos 2y = \sqrt{1 - \frac{x^2}{16}}$

Question 14 continued

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{8\sqrt{1-\frac{1}{16}x^2}} = \frac{1}{8\sqrt{\frac{1}{16}}\sqrt{16-x^2}}$$

$$= \frac{1}{2\sqrt{16-x^2}}.$$