Answer **all** questions in the spaces provided.

1 Given that $a > 0$, determine which of these expressions is **not** equivalent to the others.

Circle your answer.

[1 mark]

$-2\log_{10}\left(\dfrac{1}{a}\right)$ $\quad$ $2\log_{10}(a)$ $\quad$ $\log_{10}(a^2)$ $\quad$ $-4\log_{10}(\sqrt{a})$

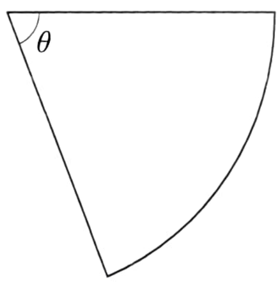
2 Given $y = e^{kx}$, where k is a constant, find $\dfrac{dy}{dx}$

Circle your answer.

[1 mark]

$\dfrac{dy}{dx} = e^{kx}$ $\quad$ $\dfrac{dy}{dx} = ke^{kx}$ $\quad$ $\dfrac{dy}{dx} = kxe^{kx-1}$ $\quad$ $\dfrac{dy}{dx} = \dfrac{e^{kx}}{k}$

3 The diagram below shows a sector of a circle.



The radius of the circle is 4 cm and $\theta = 0.8$ radians.

Find the area of the sector.

Circle your answer.

[1 mark]

1.28 cm^2 $\quad$ 3.2 cm^2 $\quad$ 6.4 cm^2 $\quad$ 12.8 cm^2

Do not write outside the box

4 The point A has coordinates $(-1, a)$ and the point B has coordinates $(3, b)$

The line AB has equation $5x + 4y = 17$

Find the equation of the perpendicular bisector of the points A and B.

[4 marks]

Midpoint: $x = 1$.

$4y = -5x + 17$

$\Rightarrow m_{perp^r} = 4/5$.

$-5 + 4a = 17 \Rightarrow a = 11/2$

$15 + 4b = 17 \Rightarrow b = \frac{1}{2}$

$\Rightarrow$ midpoint of AB is $(1, 3)$

$y - 3 = 4/5 (x - 1)$

$\Rightarrow 5y - 15 = 4x - 4$

$\Rightarrow 5y = 4x + 11$

Turn over for the next question

Do not write outside the box

5 An arithmetic sequence has first term a and common difference d.

The sum of the first 16 terms of the sequence is 260

5 (a) Show that $4a + 30d = 65$

[2 marks]

$S_{16} = 260.$

$\frac{16}{2}(2a + 15d) = 260$

$\Rightarrow 2a + 15d = 32.5$

$\Rightarrow 4a + 30d = 65$

5 (b) Given that the sum of the first 60 terms is 315, find the sum of the first 41 terms.

[3 marks]

$S_{60} = 315 \qquad 30(2a + 59d) = 315$

$\Rightarrow 2a + 59d = 10.5$

$\Rightarrow 4a + 118d = 21$

$\Rightarrow 88d = -44$

$\Rightarrow d = -\frac{1}{2}$

$\Rightarrow a = 20$

$S_{41} = \frac{41}{2}(40 - 20) = 410.$

Do not write outside the box

5 (c) S_n is the sum of the first n terms of the sequence.

Explain why the value you found in part **(b)** is the maximum value of S_n

[2 marks]

When $n = 41$, $U_{41} = 0$.

So, for $n > 41$, $U_n < 0$

and for $n < 41$, $U_n > 0$.

The value of S_n begins to decrease for $n > 41$, so it is the maximum value.

Turn over for the next question

Do not write outside the box

6 The function f is defined by

$$f(x) = \frac{1}{2}(x^2 + 1),\ x \geq 0$$

6 (a) Find the range of f.

[1 mark]

$x \geqslant \frac{1}{2}$

6 (b) (i) Find $f^{-1}(x)$

[3 marks]

$y = \frac{1}{2}(x^2+1) \Rightarrow \quad x = \frac{1}{2}(y^2+1)$

$\Rightarrow \quad 2x = y^2 + 1$

$\Rightarrow y^2 = 2x - 1$

$\Rightarrow y = \sqrt{2x-1}$

$\Rightarrow x \geqslant \frac{1}{2}$

6 (b) (ii) State the range of $f^{-1}(x)$

[1 mark]

$f^{-1}(x) \geq 0.$

Do not write outside the box

6 (c) State the transformation which maps the graph of $y = f(x)$ onto the graph of $y = f^{-1}(x)$

[1 mark]

Reflection in $y = x$.

6 (d) Find the coordinates of the point of intersection of the graphs of $y = f(x)$ and $y = f^{-1}(x)$

[2 marks]

$\frac{1}{2}(x^2+1) = \sqrt{2x-1}$

$\frac{1}{4}(x^4+2x^2+1) - 2x + 1 = 0$.

$\Rightarrow x^4 + 2x^2 - 8x + 5 = 0$

$\Rightarrow x = 1$

$\Rightarrow y = 1$.

Turn over for the next question

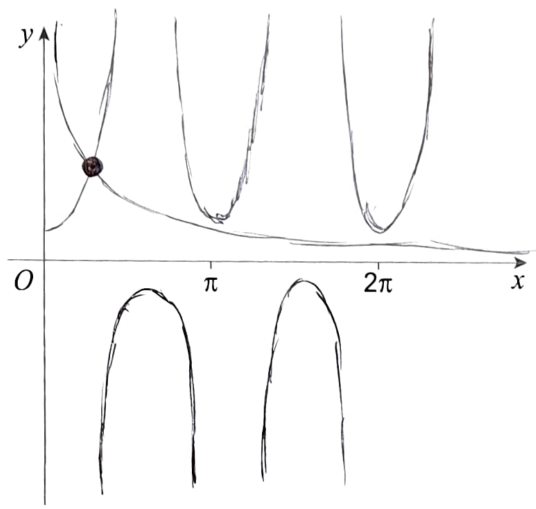
Do not write outside the box

7 (a) By sketching the graphs of $y = \frac{1}{x}$ and $y = \sec 2x$ on the axes below, show that the equation

$$\frac{1}{x} = \sec 2x$$

has exactly one solution for $x > 0$

[3 marks]



7 (b) By considering a suitable change of sign, show that the solution to the equation lies between 0.4 and 0.6

[2 marks]

$\frac{1}{x} - \sec 2x = 0$.

$\frac{1}{0.4} - \sec 0.8 = 1.06\cancel{4}...$

$\frac{1}{0.6} - \sec 1.2 = -1.093...$

The solution must lie between 0.4 and 0.6.

7 (c) Show that the equation can be rearranged to give

$$x = \frac{1}{2}\cos^{-1} x$$

[2 marks]

$\frac{1}{x} = \frac{1}{\cos 2x} \Rightarrow x = \cos 2x$

$\Rightarrow x = \frac{1}{2}\cos^{-1} x$.

Do not write outside the box

7 (d) (i) Use the iterative formula

$$x_{n+1} = \frac{1}{2}\cos^{-1} x_n$$

with $x_1 = 0.4$, to find x_2, x_3 and x_4, giving your answers to four decimal places.

[2 marks]

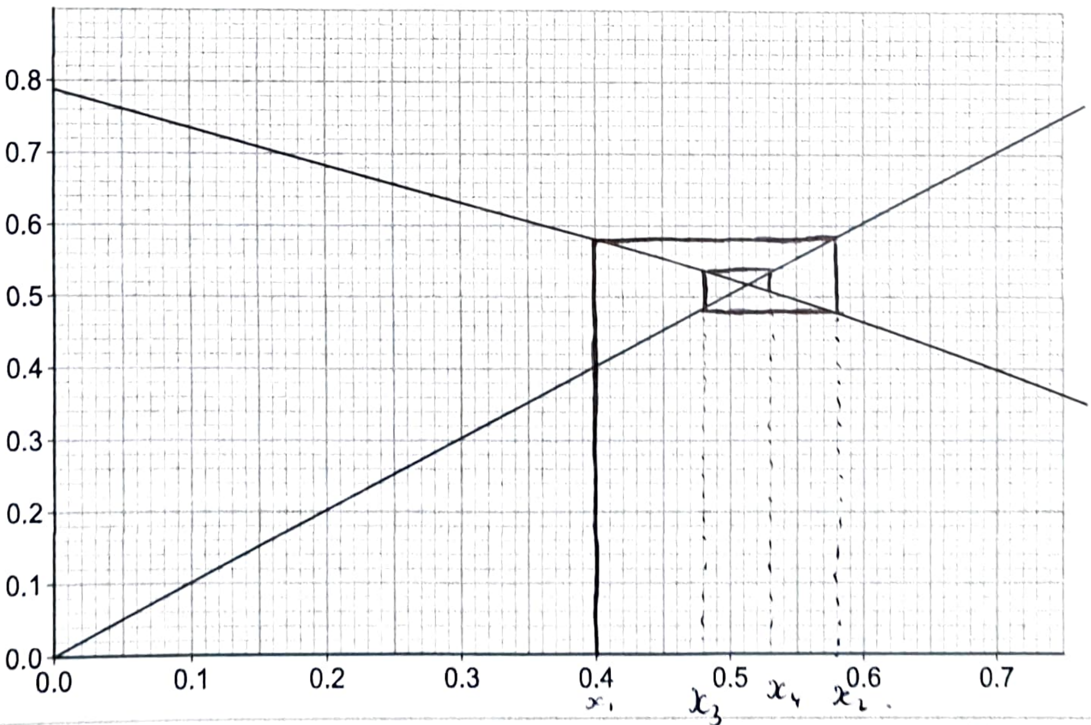
$x_2 = \frac{1}{2}\cos^{-1} 0.4 = 0.5796$

$x_3 = \frac{1}{2}\cos^{-1} 0.5796\ldots = 0.4763\ldots$

$x_4 = \frac{1}{2}\cos^{-1} 0.4763\ldots = 0.5372$

7 (d) (ii) On the graph below, draw a cobweb or staircase diagram to show how convergence takes place, indicating the positions of x_2, x_3 and x_4.

[2 marks]



Turn over ▶

Do not write outside the box

8 $P(n) = \sum_{k=0}^{n} k^3 - \sum_{k=0}^{n-1} k^3$ where n is a positive integer.

8 (a) Find $P(3)$ and $P(10)$

[2 marks]

$P(n) = n^3$

$\Rightarrow P(3) = 27$

$\Rightarrow P(10) = 1000$

8 (b) Solve the equation $P(n) = 1.25 \times 10^8$

[2 marks]

$P(n) = n^3 = 1.25 \times 10^8$

$\Rightarrow n = 500.$

Do not write outside the box

9 Prove that the sum of a rational number and an irrational number is always irrational.

[5 marks]

Assume their sum is rational.

Then let the rational number be given by a/b, and the irrational number by n. Also, let their sum be given by c/d.

a, b, c and d are all integers.

$$\frac{a}{b} + n = \frac{c}{d}$$

Then $n = \frac{c}{d} - \frac{a}{b}$

$$= \frac{bc - ad}{bd}$$

Then ~~[illegible]~~ n is rational, which contradicts the assumption that n is irrational.

Turn over for the next question

Do not write outside the box

10 The volume of a spherical bubble is increasing at a constant rate.

Show that the rate of increase of the radius, r, of the bubble is inversely proportional to r^2

$$\text{Volume of a sphere} = \frac{4}{3}\pi r^3$$

[4 marks]

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$. $\frac{dV}{dt} = k(>0)$

$\frac{dr}{dt} = \frac{dV}{dt} \div \frac{dV}{dr}$

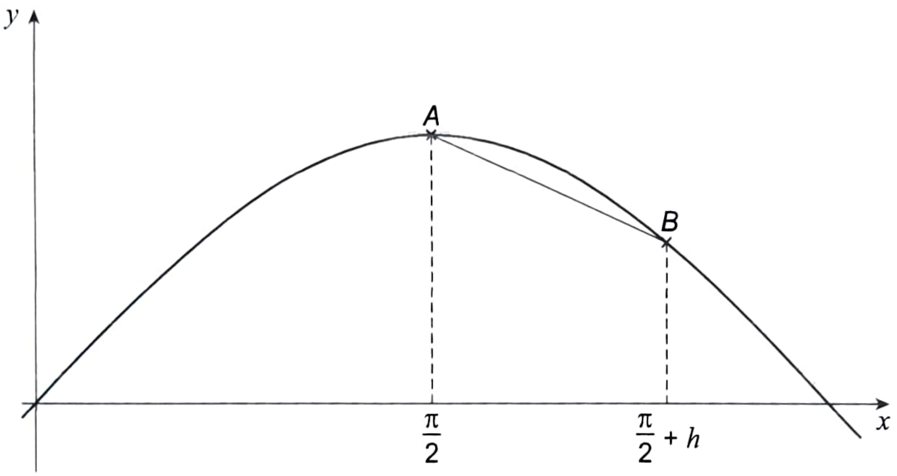
$= \frac{k}{4\pi r^2}$

$\Rightarrow \frac{dr}{dt} \propto \frac{1}{r^2}$.

Do not write outside the box

11 Jodie is attempting to use differentiation from first principles to prove that the gradient of $y = \sin x$ is zero when $x = \frac{\pi}{2}$

Jodie's teacher tells her that she has made mistakes starting in Step 4 of her working. Her working is shown below.



Step 1 Gradient of chord $AB = \dfrac{\sin\left(\frac{\pi}{2} + h\right) - \sin\left(\frac{\pi}{2}\right)}{h}$

Step 2 $= \dfrac{\sin\left(\frac{\pi}{2}\right)\cos(h) + \cos\left(\frac{\pi}{2}\right)\sin(h) - \sin\left(\frac{\pi}{2}\right)}{h}$

Step 3 $= \sin\left(\frac{\pi}{2}\right)\left(\dfrac{\cos(h) - 1}{h}\right) + \cos\left(\frac{\pi}{2}\right)\dfrac{\sin(h)}{h}$

Step 4 For gradient of curve at A,

let $h = 0$ then

$\dfrac{\cos(h) - 1}{h} = 0$ and $\dfrac{\sin(h)}{h} = 0$

Step 5 Hence the gradient of the curve at A is given by

$\sin\left(\frac{\pi}{2}\right) \times 0 + \cos\left(\frac{\pi}{2}\right) \times 0 = 0$

Do not write outside the box

Complete Steps 4 and 5 of Jodie's working below, to correct her proof.

[4 marks]

Step 4 For gradient of curve at *A*,

let $h \to 0$. Then $\frac{\cos(h)-1}{h} \to 0$

and $\frac{\sin(h)}{h} \to 0$.

Step 5 Hence the gradient of the curve at *A* is given by

$\sin\left(\frac{\pi}{2}\right) \times 0 + \cos\left(\frac{\pi}{2}\right) \times 1$

Turn over for the next question

$\frac{\sin(h)}{h} = 1$.

Do not write outside the box

12 (a) Show that the equation

$$2\cot^2 x + 2\operatorname{cosec}^2 x = 1 + 4\operatorname{cosec} x$$

can be written in the form

$$a\operatorname{cosec}^2 x + b\operatorname{cosec} x + c = 0$$

[2 marks]

$2(\operatorname{cosec}^2 x - 1) + 2\operatorname{cosec}^2 x = 1 + 4\operatorname{cosec} x$

$\Rightarrow \quad 4\operatorname{cosec}^2 x - 4\operatorname{cosec} x - 3 = 0$

Do not write outside the box

12 (b) Hence, given x is obtuse and

$$2\cot^2 x + 2\operatorname{cosec}^2 x = 1 + 4\operatorname{cosec} x$$

find the exact value of $\tan x$

Fully justify your answer.

[5 marks]

$4\operatorname{cosec}^2 x - 4\operatorname{cosec} x - 3 = 0$

$\Rightarrow \operatorname{cosec} x = \dfrac{4 \pm \sqrt{16+48}}{8} = \frac{3}{2}, \frac{-1}{2}$

Since $|\operatorname{cosec} x| \geq 1$, $\operatorname{cosec} x \neq \frac{-1}{2}$

$\Rightarrow \operatorname{cosec}^2 x = \frac{9}{4}$.

$2\cot^2 x + \frac{9}{2} = 1 + 6$

$\Rightarrow 2\cot^2 x = \frac{5}{2}$

$\Rightarrow \cot^2 x = \frac{5}{4}$

$\Rightarrow \tan^2 x = \frac{4}{5}$

$\Rightarrow \tan x = \frac{-2}{\sqrt{5}}$.

Turn over for the next question

Do not write outside the box

13 A curve, C, has equation

$$y = \frac{e^{3x-5}}{x^2}$$

Show that C has exactly one stationary point.

Fully justify your answer.

[7 marks]

$f = e^{3x-5} \Rightarrow f' = 3e^{3x-5}$

$g = x^2 \Rightarrow g' = 2x.$

$\Rightarrow \frac{dy}{dx} = \frac{3x^2e^{3x-5} - 2xe^{3x-5}}{x^4}$

$= \frac{(3x^2 - 2x)e^{3x-5}}{x^4}$

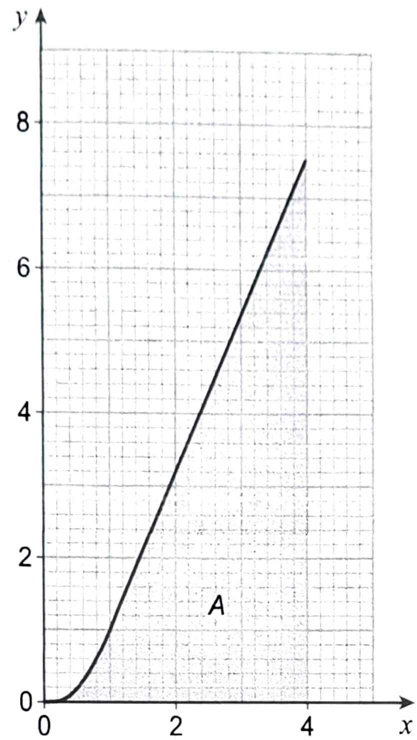
$= \frac{x(3x-2)e^{3x-5}}{x^4}$

Stationary points occur when $\frac{dy}{dx} = 0$.

$x \neq 0$ as the denominator is zero in this case.

$e^{3x-5} \neq 0$, so the only solution is $3x - 2 = 0$, or, $x = \frac{2}{3}$.

14 The graph of $y = \dfrac{2x^3}{x^2 + 1}$ is shown for $0 \le x \le 4$



Caroline is attempting to approximate the shaded area, A, under the curve using the trapezium rule by splitting the area into n trapezia.

14 (a) When $n = 4$

14 (a) (i) State the number of ordinates that Caroline uses.

[1 mark]

5.

14 (a) (ii) Calculate the area that Caroline should obtain using this method.

Give your answer correct to two decimal places.

[3 marks]

x	0	1	2	3	4
y	0	1	3.2	5.4	7.5294

Do not write outside the box

$\frac{1}{2} \times 1 \times (0 + 7.5294 + 2(1 + 3.2 + 5.4))$

$= 13.36$

14 (b) Show that the exact area of A is

$$16 - \ln 17$$

Fully justify your answer.

[5 marks]

Let $u = x^2 + 1$. Then $\frac{du}{dx} = 2x$.

$x = 0 \Rightarrow u = 1$ Then $dx = \frac{du}{2x}$.

$x = 4 \Rightarrow u = 17$.

$\int \frac{2x^3}{u} \frac{du}{2x} = \int_1^{17} \frac{x^2}{u} du$

$= \int_1^{17} \frac{u-1}{u} du$

$= \int_1^{17} 1 - \frac{1}{u} du$

$= [u - \ln u]_1^{17}$.

$= 17 - \ln 17 - 1$

$= 16 - \ln 17$.

Question 14 continues on the next page

Do not write outside the box

14 (c) Explain what would happen to Caroline's answer to part **(a)(ii)** as $n \to \infty$

[1 mark]

The approximation tends towards
$16 - \ln 17$.

Do not write outside the box

15 (a) At time t hours **after a high tide**, the height, h metres, of the tide and the velocity, v knots, of the tidal flow can be modelled using the parametric equations

$$v = 4 - \left(\frac{2t}{3} - 2\right)^2$$

$$h = 3 - 2\sqrt[3]{t - 3}$$

High tides and low tides occur alternately when the velocity of the tidal flow is zero.

A high tide occurs at 2 am.

15 (a) (i) Use the model to find the height of this high tide.

[1 mark]

$h = 3 - 2\sqrt[3]{-3} = 5.88\text{m}.$

15 (a) (ii) Find the time of the first **low** tide after 2 am.

[3 marks]

$V = 0 = 4 - \left(\frac{2t}{3} - 2\right)^2$

$\Rightarrow \frac{2t}{3} - 2 = 2.$

$\Rightarrow t = 6.$

$\Rightarrow$ 8am

15 (a) (iii) Find the height of this low tide.

[1 mark]

$h = 3 - 2\sqrt[3]{3} = 0.12\text{m}.$

Do not write outside the box

15 (b) Use the model to find the height of the tide when it is flowing with maximum velocity.

[3 marks]

v is maximised when $\frac{2t}{3} - 2 = 0$,

or, $t = 3$.

When $t=3$,

$h = 3 - 2\sqrt[3]{0} = 3\text{m}$.

15 (c) Comment on the validity of the model.

[2 marks]

The model is limited by time.

As t increases, the height continues to decrease (i.e. after the point of low tide). (When $t=6$).

Turn over for the next question

Do not write outside the box

16 (a) $y = e^{-x}(\sin x + \cos x)$

Find $\dfrac{dy}{dx}$

Simplify your answer.

[3 marks]

$y = e^{-x}(\sin x + \cos x)$

$f = e^{-x}$

$f' = -e^{-x}$

$\dfrac{dy}{dx} = -e^{-x}(\sin x + \cos x) + e^{-x}(\cos x - \sin x)$

$g = \sin x + \cos x$

$= -2e^{-x} \sin x .$

$g' = \cos x - \sin x.$

16 (b) Hence, show that

$$\int e^{-x} \sin x \, dx = ae^{-x}(\sin x + \cos x) + c$$

where a is a rational number.

[2 marks]

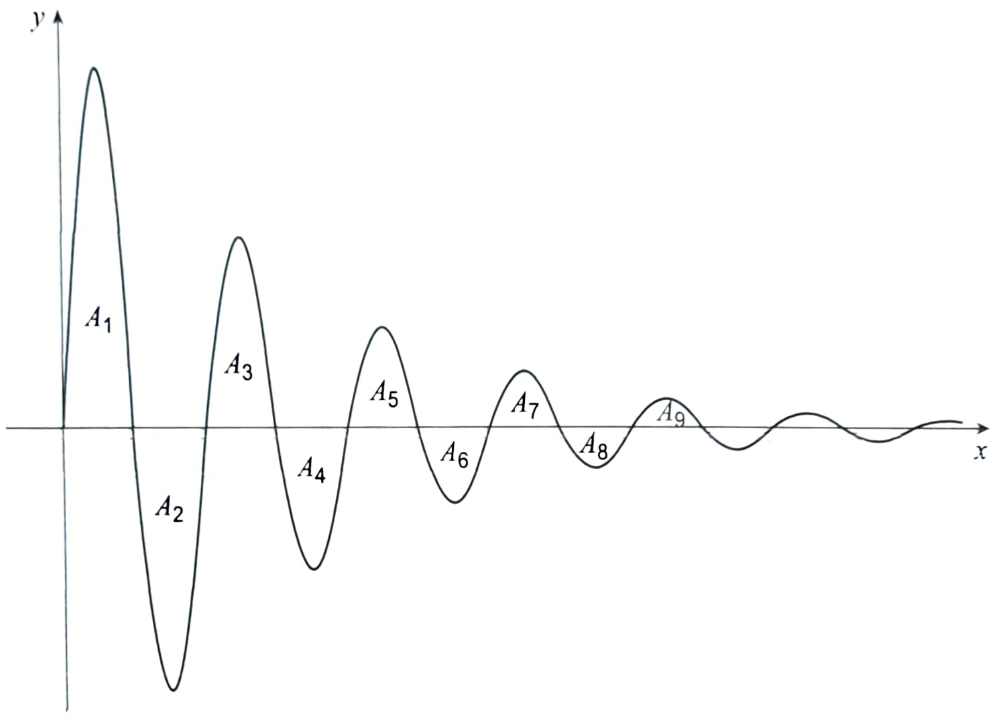
$\int -2e^{-x} \sin x \, dx = e^{-x}(\sin x + \cos x) + c$

$\Rightarrow \int e^{-x} \sin x \, dx = \frac{-1}{2} e^{-x}(\sin x + \cos x) + c.$

Do not write outside the box

16 (c) A sketch of the graph of $y = e^{-x} \sin x$ for $x \geq 0$ is shown below.

The areas of the finite regions bounded by the curve and the x-axis are denoted by $A_1, A_2, \ldots, A_n, \ldots$



16 (c) (i) Find the exact value of the area A_1

[3 marks]

Limits of A_1 are 0 and π.

$$\int_0^{\pi} e^{-x} \sin x \, dx = -\frac{1}{2}\left[e^{-x}(\sin x + \cos x)\right]_0^{\pi}$$

$$= -\frac{1}{2}\left[-e^{-\pi} - 1\right]$$

$$= \frac{e^{-\pi} + 1}{2}$$

Question 16 continues on the next page

Do not write outside the box

16 (c) (ii) Show that

$$\frac{A_2}{A_1} = e^{-\pi}$$

[4 marks]

Limits of A_2 are π and 2π.

$$A_2 = \int_{\pi}^{2\pi} y \, dx = -\frac{1}{2}\left[e^{-x}(\sin x + \cos x)\right]_{\pi}^{2\pi}$$

$$= -\frac{1}{2}\left[e^{-2\pi} + e^{-\pi}\right]$$

$$= \frac{e^{-\pi} + e^{-2\pi}}{2} = \frac{e^{-\pi}+1}{2} e^{-\pi}.$$

$$\frac{A_2}{A_1} = \frac{\frac{e^{-\pi}+1}{2} e^{-\pi}}{\frac{e^{-\pi}+1}{2}} = e^{-\pi}$$

Do not write outside the box

16 (c) (iii) Given that

$$\frac{A_{n+1}}{A_n} = e^{-\pi}$$

show that the exact value of the total area enclosed between the curve and the x-axis is

$$\frac{1 + e^{\pi}}{2(e^{\pi} - 1)}$$

[4 marks]

The values of A_n form a geometric series.

$$S_\infty = \frac{a}{1-r} = \frac{e^{-\pi}+1}{2} \times \frac{1}{1-e^{-\pi}}$$

$$= \frac{1 + e^{-\pi}}{2(e^{-\pi} - 1)}$$

END OF QUESTIONS