AQA

Please write clearly in block capitals.

Centre number | | | | | | Candidate number | | | | |

Surname MODEL SOLUTIONS

Forename(s) ______

Candidate signature ______

A-level
MATHEMATICS
Paper 1

Wednesday 6 June 2018 Morning Time allowed: 2 hours

Materials
- You must have the AQA Formulae for A-level Mathematics booklet.
- You should have a graphical or scientific calculator that meets the requirements of the specification.

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- You must answer each question in the space provided for that question. If you require extra space, use an AQA supplementary answer book; do **not** use the space provided for a different question.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work that you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 100.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.
- You do not necessarily need to use all the space provided.

For Examiner's Use	
Question	Mark
1	
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TOTAL	

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Answer **all** questions in the spaces provided.

1 $y = \dfrac{1}{x^2}$

Find an expression for $\dfrac{dy}{dx}$

Circle your answer.

[1 mark]

$\dfrac{dy}{dx} = \dfrac{0}{2x}$ $\qquad$ $\dfrac{dy}{dx} = x^{-2}$ $\qquad$ $\dfrac{dy}{dx} = -\dfrac{2}{x}$ $\qquad$ $\dfrac{dy}{dx} = -\dfrac{2}{x^3}$

2 The graph of $y = 5^x$ is transformed by a stretch in the y-direction, scale factor 5

State the equation of the transformed graph.

Circle your answer.

[1 mark]

$y = 5 \times 5^x$ $\qquad$ $y = 5^{\frac{x}{5}}$ $\qquad$ $y = \dfrac{1}{5} \times 5^x$ $\qquad$ $y = 5^{5x}$

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3 A periodic sequence is defined by $U_n = \sin\left(\frac{n\pi}{2}\right)$

State the period of this sequence.

Circle your answer.

[1 mark]

8 2π 4 π

4 The function f is defined by $\mathrm{f}(x) = \mathrm{e}^{x-4}, x \in \mathbb{R}$

Find $\mathrm{f}^{-1}(x)$ and state its domain.

[3 marks]

$f(x) = e^{x-4}$

$y = e^{x-4}$

$\ln(y) = x - 4$

$x = \ln(y) + 4$

$\Rightarrow f^{-1}(x) = \ln(x) + 4$.

Turn over for the next question

Turn over ▶

Do not write outside the box

5 A curve is defined by the parametric equations

$$x = 4 \times 2^{-t} + 3$$

$$y = 3 \times 2^{t} - 5$$

5 (a) Show that $\frac{dy}{dx} = -\frac{3}{4} \times 2^{2t}$

[3 marks]

$\frac{dy}{dt} = 3\ln(2) \times 2^{t}$

$\frac{dx}{dt} = -4\ln(2) \times 2^{-t}$

$\frac{dy}{dx} = \frac{3\ln 2 \times 2^{t}}{-4\ln 2 \times 2^{-t}}$

$= -\frac{3}{4} \times 2^{2t}$

5 (b) Find the Cartesian equation of the curve in the form $xy + ax + by = c$, where a, b and c are integers.

[3 marks]

$x = 4 \times 2^{-t} + 3$ $\quad\quad$ $y = 3 \times 2^{t} - 5$

$x - 3 = 4 \times 2^{-t}$ $\quad\quad$ $y + 5 = 3 \times 2^{t}$

$\frac{x-3}{4} = 2^{-t}$ $\quad\quad$ $\frac{y+5}{3} = 2^{t}$.

$\left(\frac{x-3}{4}\right)\left(\frac{y+5}{3}\right) = 1$.

$(x-3)(y+5) = 12$.

$xy + 5x - 3y - 15 = 12$

$xy + 5x - 3y = 27$

Do not write outside the box

6 (a) Find the first three terms, in ascending powers of x, of the binomial expansion of $\frac{1}{\sqrt{4+x}}$

[3 marks]

$$\frac{1}{\sqrt{4+x}} = (4+x)^{-\frac{1}{2}}$$
$$= \frac{1}{2}\left(1+\frac{1}{4}x\right)^{-\frac{1}{2}}$$
$$= \frac{1}{2}\left(1+\left(-\frac{1}{2}\right)\left(\frac{x}{4}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{x}{4}\right)^2}{2}\right)$$
$$= \frac{1}{2}\left(1-\frac{x}{8}+\frac{3x^2}{128}\right)$$
$$= \frac{1}{2}-\frac{1}{16}x+\frac{3}{256}x^2$$

6 (b) Hence, find the first three terms of the binomial expansion of $\frac{1}{\sqrt{4-x^3}}$

[2 marks]

$$= \frac{1}{2}+\frac{1}{16}x^3+\frac{3}{256}x^6.$$
$$(x \to -x^3)$$

Question 6 continues on the next page

Turn over ▶

Do not write outside the box

6 (c) Using your answer to part **(b)**, find an approximation for $\int_0^1 \frac{1}{\sqrt{4-x^3}}\,dx$, giving your answer to seven decimal places.

[3 marks]

$\int_0^1 \frac{1}{\sqrt{4-x^3}}\,dx =$

$\int_0^1 \frac{1}{2} + \frac{1}{16}x^3 + \frac{3}{256}x^6\,dx =$

$\left[\frac{1}{2}x + \frac{1}{64}x^4 + \frac{3}{1792}x^7\right]_0^1 =$

$\frac{1}{2} + \frac{1}{64} + \frac{3}{1792} =$

0.5172291

6 (d) (i) Edward, a student, decides to use this method to find a more accurate value for the integral by increasing the number of terms of the binomial expansion used.

Explain clearly whether Edward's approximation will be an overestimate, an underestimate, or if it is impossible to tell.

[2 marks]

Each term is positive, so we are always increasing towards the actual value, so Edward's approximation will be an underestimate.

Do not write outside the box

6 (d) (ii) Edward goes on to use the expansion from part **(b)** to find an approximation for $\int_{-2}^{0} \frac{1}{\sqrt{4-x^3}}\, dx$

Explain why Edward's approximation is invalid.

[2 marks]

The binomial expression is valid for

$|x^3| < 4$

$|x| < 4^{\frac{1}{3}}$

$2 > 4^{\frac{1}{3}}$

so -2 is not in range.

Turn over for the next question

7 Three points A, B and C have coordinates A (8, 17), B (15, 10) and C (−2, −7)

7 (a) Show that angle ABC is a right angle.

[3 marks]

$AB^2 = (15-8)^2 + (17-10)^2 = 7^2 + 7^2 = 49 + 49 = 98$

$BC^2 = (-2-15)^2 + (-7-10)^2 = 17^2 + 17^2 = 289 + 289 = 578$

$AC^2 = (-2-8)^2 + (-7-17)^2 = 10^2 + 24^2 = 100 + 576 = 676$

$578 + 98 = 676$

Hence, ABC is a right angled triangle.

7 (b) A, B and C lie on a circle.

7 (b) (i) Explain why AC is a diameter of the circle.

[1 mark]

The angle subtended by a diameter is 90°, and ABC is a right-angle triangle. Hence, AC must be a diameter.

7 (b) (ii) Determine whether the point D $(-8, -2)$ lies inside the circle, on the circle or outside the circle.

Fully justify your answer.

[4 marks]

A (8, 7) C (−2, −7)

⇒ Centre is (3, 5)

Radius is $\frac{\sqrt{676}}{2} = \frac{26}{2} = 13 = \sqrt{169}$

Distance from centre to (−8, −2):

$\sqrt{(-8-3)^2 + (-2-5)^2} =$

$\sqrt{11^2 + 7^2} =$

$\sqrt{121 + 49} =$

$\sqrt{170} > \sqrt{169}$.

Hence, D is not in the circle.

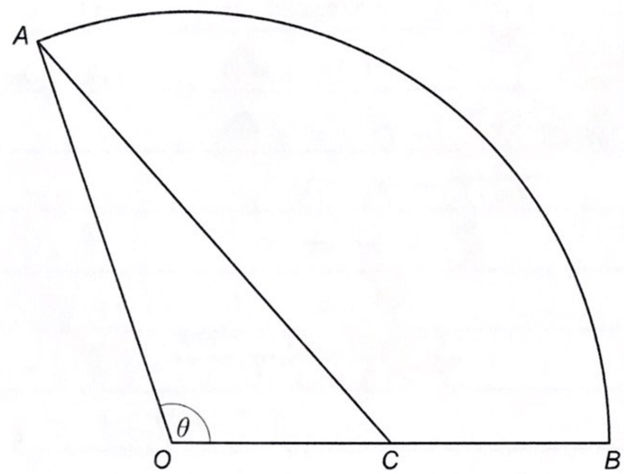
Turn over for the next question

Do not write outside the box

8 The diagram shows a sector of a circle OAB.

C is the midpoint of OB.

Angle AOB is θ radians.



8 (a) Given that the area of the triangle OAC is equal to one quarter of the area of the sector OAB, show that $\theta = 2\sin\theta$

[4 marks]

Triangle area $= \frac{1}{2}ab\sin C$

Use r to denote radius of circle.

$OA = r$, $OC = \frac{1}{2}r$, $OB = r$.

Triangle area $= \frac{1}{2}r\left(\frac{1}{2}r\right)\sin\theta$

$= \frac{1}{4}r^2\sin\theta$.

Circle area $= \frac{1}{2}r^2\theta$.

$\frac{1}{4} \times \frac{1}{2}r^2\theta = \frac{1}{4}r^2\sin\theta$

$\frac{1}{8}r^2\theta = \frac{1}{4}r^2\sin\theta$

$2\sin\theta = \theta$.

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8 (b) Use the Newton-Raphson method with $\theta_1 = \pi$, to find θ_3 as an approximation for θ. Give your answer correct to five decimal places.

[3 marks]

$f(\theta) = \theta - 2\sin\theta = 0$.

$f'(\theta) = 1 - 2\cos\theta$.

Iteration: $\theta_{n+1} = \theta_n - \dfrac{\theta_n - 2\sin\theta_n}{1 - 2\cos\theta_n}$

$\theta_2 = \pi - \dfrac{\pi - 2\sin\pi}{1 - 2\cos\pi}$

$= \pi - \dfrac{\pi}{1+2}$

$= \pi - \dfrac{\pi}{3}$

$= \dfrac{2\pi}{3}$

$\theta_3 = \dfrac{2\pi}{3} - \dfrac{2\pi/3 - 2\sin(2\pi/3)}{1 - 2\cos(2\pi/3)}$

$\theta_3 = 1.91322$

8 (c) Given that $\theta = 1.89549$ to five decimal places, find an estimate for the percentage error in the approximation found in part **(b)**.

[1 mark]

error $= \dfrac{1.91322 - 1.89549}{1.91322}$

error $= 0.935\%$.

Turn over for the next question

Do not write outside the box

9 An arithmetic sequence has first term a and common difference d.

The sum of the first 36 terms of the sequence is equal to the square of the sum of the first 6 terms.

9 (a) Show that $4a + 70d = 4a^2 + 20ad + 25d^2$

[4 marks]

$S_6 = 3(2a+5d) = 6a+15d$

$S_{36} = 18(2a+35d) = 36a+630d$

$36a + 630d = (6a+15d)^2$

$36a + 630d = 36a^2 + 180ad + 225d^2$

$4a + 70d = 4a^2 + 20ad + 25d^2$

Do not write outside the box

9 (b) Given that the sixth term of the sequence is 25, find the smallest possible value of a.

[5 marks]

$a + 5d = 25 \Rightarrow d = \frac{25-a}{5}$

$4a + 70\left(\frac{25-a}{5}\right) = 4a^2 + 20a\left(\frac{25-a}{5}\right) + 25\left(\frac{25-a}{5}\right)^2$

$4a + 14(25-a) = 4a^2 + 4a(25-a) + (25-a)^2$

$4a + 350 - 14a = 4a^2 + 100a - 4a^2 + a^2 - 50a + 625$

$350 - 10a = a^2 + 50a + 625$

$a^2 + 60a + 275 = 0$.

$(a+5)(a+55) = 0$

$a = -5 \quad a = -55$

$\Rightarrow$ Minimum is $a = -55$

Turn over for the next question

Do not write outside the box

10 A scientist is researching the effects of caffeine. She models the mass of caffeine in the body using

$$m = m_0 e^{-kt}$$

where m_0 milligrams is the initial mass of caffeine in the body and m milligrams is the mass of caffeine in the body after t hours.

On average, it takes 5.7 hours for the mass of caffeine in the body to halve.

One cup of strong coffee contains 200 mg of caffeine.

10 (a) The scientist drinks two strong cups of coffee at 8 am. Use the model to estimate the mass of caffeine in the scientist's body at midday.

[4 marks]

$m = m_0 e^{-kt}$

$\frac{1}{2} m_0 = m_0 e^{-5.7k}$

$\frac{1}{2} = e^{-5.7k}$

$\log\left(\frac{1}{2}\right) = -5.7k$.

$\log(2) = 5.7k$.

$k = \frac{\log(2)}{5.7}$

$m = 400 e^{-4 \times \frac{\log(2)}{5.7}}$

$m = 245.9 \text{mg}$

$m = 250 \text{ mg}$ estimate.

Do not write outside the box

10 (b) The scientist wants the mass of caffeine in her body to stay below 480 mg

Use the model to find the earliest time that she could drink another cup of strong coffee.

Give your answer to the nearest minute.

[3 marks]

$400e^{-\frac{\log 2}{5.7}t} < 280$

$e^{-\frac{\log 2}{5.7}t} < 0.7$

$-\frac{\log 2}{5.7}t < \log(0.7)$

$t > \frac{-5.7\log(0.7)}{\log(2)}$

$t > 2.933$

2h 56m.

10:56 am

10 (c) State a reason why the mass of caffeine remaining in the scientist's body predicted by the model may not be accurate.

[1 mark]

Different people process caffeine at different rates.

Turn over for the next question

Do not write outside the box

11 The daily world production of oil can be modelled using

$$V = 10 + 100\left(\frac{t}{30}\right)^3 - 50\left(\frac{t}{30}\right)^4$$

where V is volume of oil in millions of barrels, and t is time in years since 1 January 1980.

11 (a) (i) The model is used to predict the time, T, when oil production will fall to zero.

Show that T satisfies the equation

$$T = \sqrt[3]{60T^2 + \frac{162\,000}{T}}$$

[3 marks]

$10 + 100\left(\frac{T}{30}\right)^3 - 50\left(\frac{T}{30}\right)^4 = 0$

$50\left(\frac{T}{30}\right)^4 = 10 + 100\left(\frac{T}{30}\right)^3$

$\frac{50T^4}{810000} = 10 + \frac{100T^3}{27000}$

$\frac{T^4}{16200} = 10 + \frac{T^3}{270}$

$T^4 = 162000 + 60T^3$

$T^3 = \frac{162000}{T} + 60T^2$

$T = \sqrt[3]{60T^2 + \frac{162000}{T}}$

11 (a) (ii) Use the iterative formula $T_{n+1} = \sqrt[3]{60{T_n}^2 + \frac{162\,000}{T_n}}$, with $T_0 = 38$, to find the values of T_1, T_2, and T_3, giving your answers to three decimal places.

[2 marks]

$T_1 = \sqrt[3]{60 \times 38^2 + \frac{162000}{38}}$

$T_1 = 44.963$

$T_2 = \sqrt[3]{60 \times 44.963^2 + \frac{162000}{44.963}}$

$T_2 = 49.987$

$T_3 = \sqrt[3]{60 \times 49.987^2 + \frac{162000}{49.987}}$

$T_3 = 53.504$

Do not write outside the box

11 (a) (iii) Explain the relevance of using $T_0 = 38$

[1 mark]

38 represents the year 2018 (current year when exam paper was first published).

11 (b) From 1 January 1980 the daily use of oil by one technologically developing country can be modelled as

$$V = 4.5 \times 1.063^t$$

Use the models to show that the country's use of oil and the world production of oil will be equal during the year 2029.

[4 marks]

$10+100\left(\frac{t}{30}\right)^3 - 50\left(\frac{t}{30}\right)^4 = 4.5 \times 1.063^t$

LHS: RHS:

$t=49$ gives 89.89 $t=49$ gives 89.81

$t=50$ gives 87.16 $t=50$ gives 95.47

So $t \in [49, 50]$

Hence, the year 2029.

Turn over for the next question

Do not write outside the box

12 $p(x) = 30x^3 - 7x^2 - 7x + 2$

12 (a) Prove that $(2x+1)$ is a factor of $p(x)$

[2 marks]

Factor theorem: If $2x+1$ is a factor then $p(-\frac{1}{2}) = 0$.

$p(-\frac{1}{2}) = 30(-\frac{1}{2})^3 - 7(-\frac{1}{2})^2 - 7(-\frac{1}{2}) + 2$

$= 30(-\frac{1}{8}) - 7(\frac{1}{4}) + 7(\frac{1}{2}) + 2$

$= \frac{-15}{4} - \frac{7}{4} + \frac{7}{2} + 2$

$= \frac{-15}{4} - \frac{7}{4} + \frac{14}{4} + \frac{8}{4}$

$= 0$.

12 (b) Factorise $p(x)$ completely.

[3 marks]

$p(x) = (2x+1)(15x^2 - 11x + 2)$

$p(x) = (2x+1)(5x-2)(3x-1)$

Do not write outside the box

12 (c) Prove that there are no real solutions to the equation

$$\frac{30\sec^2 x + 2\cos x}{7} = \sec x + 1$$

[5 marks]

$\frac{30\sec^2 x + 2\cos x}{7} = \sec x + 1.$

$30\sec^2 x + 2\cos x = 7\sec x + 7$

$30\sec^3 x + 2 = 7\sec^2 x + 7\sec x$

$30\sec^3 x - 7\sec^2 x - 7\sec x + 2 = 0.$

$(2\sec x + 1)(5\sec x - 2)(3\sec x - 1) = 0.$

$\sec x = -\frac{1}{2}, \frac{1}{3}, \frac{2}{5}$

None of these values fall in the range of sec x, as the function does not take values in -1, 1.

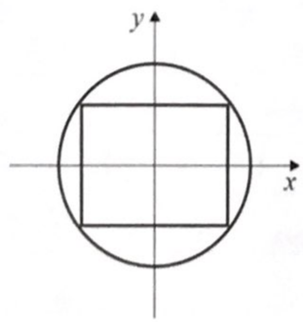
Hence, no real solutions.

Turn over for the next question

Do not write outside the box

13 A company is designing a logo. The logo is a circle of radius 4 inches with an inscribed rectangle. The rectangle must be as large as possible.

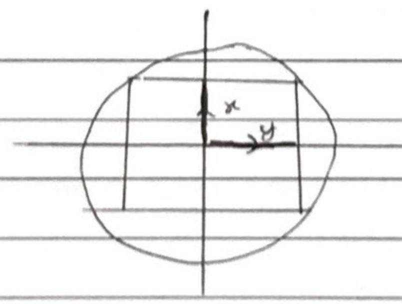
The company models the logo on an x-y plane as shown in the diagram.



Use calculus to find the maximum area of the rectangle.

Fully justify your answer.

[10 marks]



Width $= 2x$, Length $= 2y$

$A = 4xy$

$x^2 + y^2 = 16$.

$y^2 = 16 - x^2$

$A = 4x\sqrt{16 - x^2}$

$\frac{dA}{dx} = 0$ at maximum.

$4\sqrt{16 - x^2} - \frac{4x^2}{\sqrt{16 - x^2}} = 0$.

$\frac{4(16 - x^2) - 4x^2}{\sqrt{16 - x^2}} = 0$.

$\frac{64 - 8x^2}{\sqrt{16 - x^2}} = 0$.

$64 - 8x^2 = 0$

$8 - x^2 = 0$.

Do not write outside the box

$x^2 = 8$

$x = \pm 2\sqrt{2}$

+ve root only makes sense.

$x = 2\sqrt{2} \quad (= 2.83)$

At $x = 2.8$, $\frac{dA}{dx} = 0.448$

At $x = 2.9$, $\frac{dA}{dx} = -1.191$.

So $x = 2\sqrt{2}$ is a maximum.

$x^2 + y^2 = 16$.

$8 + y^2 = 16$

$y^2 = 8$

$y = 2\sqrt{2}$.

$A = 4xy$

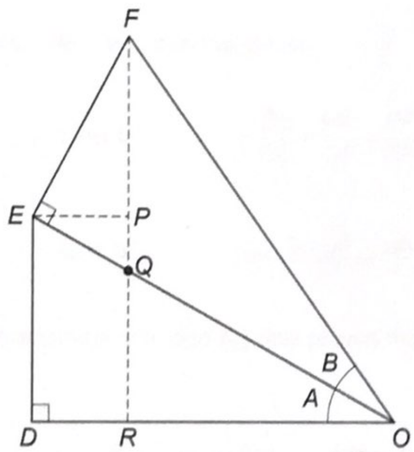
$A = 4 \times 2\sqrt{2} \times 2\sqrt{2}$

$A = 32$ square inches

Turn over for the next question

14 Some students are trying to prove an identity for $\sin(A+B)$.

They start by drawing two right-angled triangles *ODE* and *OEF*, as shown.



The students' incomplete proof continues,

Let angle $DOE = A$ and angle $EOF = B$.

In triangle *OFR*,

Line 1 $\sin(A+B) = \dfrac{RF}{OF}$

Line 2 $= \dfrac{RP + PF}{OF}$

Line 3 $= \dfrac{DE}{OF} + \dfrac{PF}{OF}$ since $DE = RP$

Line 4 $= \dfrac{DE}{\dots\dots} \times \dfrac{\dots\dots}{OF} + \dfrac{PF}{EF} \times \dfrac{EF}{OF}$

Line 5 $= \dots\dots\dots\dots\dots + \cos A \sin B$

14 (a) Explain why $\dfrac{PF}{EF} \times \dfrac{EF}{OF}$ in Line 4 leads to $\cos A \sin B$ in Line 5

[2 marks]

$\angle OQR = \angle FQE$ vertically opposite.

$\angle ORQ = \angle FEQ = 90°$.

So $\angle EFQ = A$.

Hence $\frac{PF}{EF} = \cos(A)$.

Do not write outside the box

And, through $\hat{O}EF$, $\frac{EF}{OF} = \sin(B)$

14 (b) Complete Line 4 and Line 5 to prove the identity

Line 4 $= \frac{DE}{OE} \times \frac{OE}{OF} + \frac{PF}{EF} \times \frac{EF}{OF}$

Line 5 $= \sin A \cos B + \cos A \sin B$

[1 mark]

14 (c) Explain why the argument used in part **(a)** only proves the identity when A and B are acute angles.

[1 mark]

Proof uses right-angled triangles, where A and B have to be acute.

14 (d) Another student claims that by replacing B with $-B$ in the identity for $\sin(A+B)$ it is possible to find an identity for $\sin(A-B)$.

Assuming the identity for $\sin(A+B)$ is correct for all values of A and B, prove a similar result for $\sin(A-B)$.

[3 marks]

$\sin(A+B) = \sin A \cos B + \cos A \sin B$

$\sin(A-B) = \sin A \cos(-B) + \cos(A)\sin(-B)$

$\cos(-B) = \cos(B)$

$\sin(-B) = -\sin(B)$

$\sin(A-B) = \sin A \cos B - \cos A \sin B$

Turn over ▶

15 A curve has equation $y = x^3 - 48x$

The point A on the curve has x coordinate -4

The point B on the curve has x coordinate $-4 + h$

15 (a) Show that the gradient of the line AB is $h^2 - 12h$

[4 marks]

$A: (-4, (-4)^3 - 48(-4)) =$

$(-4, -64 + 192) =$

$(-4, 128)$

$B: (-4+h, (-4+h)^3 - 48(-4+h)) =$

$(-4+h, -64 + 48h - 12h^2 + h^3 + 192 - 48h) =$

$(-4+h, h^3 - 12h^2 + 128)$

$\text{Gradient} = \dfrac{h^3 - 12h^2 + 128 - 128}{-4 + h + 4} = \dfrac{h^3 - 12h^2}{h} = h^2 - 12h$

15 (b) Explain how the result of part **(a)** can be used to show that A is a stationary point on the curve.

[2 marks]

Gradient of curve is given by $h^2 - 12h$. As $h \to 0$, $h^2 - 12h \to 0$, so the gradient at A is 0. Hence, A is a stationary point.

END OF QUESTIONS