AQA

A Level

A Level Maths

AQA Core Maths C4 January 2013 Model Solutions

Name:

Mathsmadeeasy.co.uk

Total Marks:

AQA Jan 13 C4

1a.

$$f(x) = 2x^3 + x^2 - 8x - 7$$

$$f(-\tfrac{1}{2}) = 2(-\tfrac{1}{2})^3 + (-\tfrac{1}{2})^2 - 8(-\tfrac{1}{2}) - 7$$

$$= -\tfrac{1}{4} + \tfrac{1}{4} + 4 - 7$$

$$= -3$$

1bi.

$$g(x) = f(x) + d$$

$$= 2x^3 + x^2 - 8x + (d-7)$$

since $(2x+1)$ is a factor $g(-\tfrac{1}{2}) = 0$

$$\Rightarrow \quad d = 3$$

1bii.

$$g(x) = 2x^3 + x^2 - 8x - 4 = (2x+1)(x^2 + a)$$

Constant terms: $\quad -4 = 1a \quad \Rightarrow \quad a = -4$

$$g(x) = (2x+1)(x^2-4)$$

$$= (2x+1)(x+2)(x-2)$$

1biii.

$$\frac{g(x)}{2x^3 - 3x^2 - 2x} = \frac{\cancel{(2x+1)}(x+2)\cancel{(x-2)}}{x\cancel{(2x+1)}\cancel{(x-2)}}$$

$$= \frac{x+2}{x}$$

$$= \frac{x}{x} + \frac{2}{x}$$

$$= 1 + \frac{2}{x}$$

2a.

$$f(x) = \frac{7x-1}{(1+3x)(3-x)} = \frac{A}{3-x} + \frac{B}{1+3x}$$

$$7x-1 = A(1+3x) + B(3-x)$$

$x = 3$; $\quad 20 = 10A \Rightarrow A = 2$

$x = -1/3$; $\quad -\frac{10}{3} = \frac{10}{3}B \Rightarrow B = -1$

$$f(x) = \frac{2}{3-x} - \frac{1}{1+3x}$$

2bi.

$$f(x) = 2(3-x)^{-1} - (1+3x)^{-1}$$

$$2(3-x)^{-1} = 2\left[3(1-x/3)\right]^{-1} = 2.\,3^{-1}(1-x/3)^{-1}$$

$$= \frac{2}{3}(1-x/3)^{-1}$$

$$= \frac{2}{3}\left(1 + (-1)(-x/3) + \frac{(-1)(-2)}{2}(-x/3)^2 + \ldots\right)$$

$$= \frac{2}{3}\left(1 + \frac{1}{3}x + \frac{1}{9}x^2 + \ldots\right)$$

$$= \frac{2}{3} + \frac{2}{9}x + \frac{2}{27}x^2 + \ldots$$

$$(1+3x)^{-1} = 1 + (-1)(3x) + \frac{(-1)(-2)}{2}(3x)^2 + \ldots$$

$$= 1 - 3x + 9x^2 + \ldots$$

$$f(x) = \left(\frac{2}{3} + \frac{2}{9}x + \frac{2}{27}x^2\right) - \left(1 - 3x + 9x^2\right)$$

$$= -\frac{1}{3} + \frac{29}{9}x - \frac{241}{27}x^2$$

2bii. expansion valid for $\left|\frac{x}{3}\right| < 1$ and $|3x| < 1$

$\therefore \quad |x| < 1/3$

$0.4 > 1/3 \quad \therefore$ will not give suitable estimation.

3ai.

$3\cos x + 2\sin x = R\cos(x - \alpha)$

$3\cos x + 2\sin x = R\cos x\cos\alpha + R\sin x\sin\alpha$

$R = \sqrt{3^2+2^2} = \sqrt{13}$

$\cos x : \quad 3 = \sqrt{13}\cos\alpha$

$\alpha = \cos^{-1}\left(\frac{3}{\sqrt{13}}\right)$

$= 33.7°$ (1d.p.)

$3\cos x + 2\sin x = \sqrt{13}\cos(x - 33.7)$

3aii.

minimum value $= -\sqrt{13}$ (when $\cos(x-33.7) = -1$)

$\cos(x-33.7) = 1$

$x - 33.7 = 180$

$x = 213.7°$

3bi.

$\cot x - \sin 2x \equiv \cot x \cos 2x \qquad 0 < x < 180$

LHS $\quad \frac{\cos x}{\sin x} - 2\sin x\cos x$

$\frac{\cos x}{\sin x} - \frac{2\sin^2 x\cos x}{\sin x}$

$= \frac{\cos x - 2\sin^2 x\cos x}{\sin x}$

$= \frac{\cos x}{\sin x}(1 - 2\sin^2 x)$

$= \cot x \cos 2x \qquad (\cos 2x \equiv 1 - 2\sin^2 x)$

$=$ RHS

3bii. $\cot x \cos 2x = 0$

$\cot x = 0$ or $\cos 2x = 0$

$x = 90°$

$2x = 90°, 270°$

$x = 45°, 135°$

4ai. $x^2 - y^2 = 8$

$2x - 2y\frac{dy}{dx} = 0$

$2y\frac{dy}{dx} = 2x$

$\frac{dy}{dx} = \frac{2x}{2y} = \frac{x}{y}$ so when $x = p$ $y = q$ at (p,q)

$\frac{dy}{dx} = \frac{p}{q}$

4aii. at (p,q) $\frac{dy}{dx} = \frac{p}{q}$: $y - q = \frac{p}{q}(x - p)$

$y = q + \frac{p}{q}(x-p)$ ①

at $(p,-q)$, $\frac{dy}{dx} = -\frac{p}{q}$: $y + q = -\frac{p}{q}(x-p)$

$y = -q - \frac{p}{q}(x-p)$ ②

'① + ②' $2y = 0$

$\therefore$ intersect on x axis

4b.

$x = t + \frac{2}{t}$ $\qquad$ $y = t - \frac{2}{t}$

$x^2 = \left(t + \frac{2}{t}\right)^2$ $\qquad$ $y^2 = t^2 - 4 + \frac{4}{t^2}$

$= t^2 + 4 + \frac{4}{t^2}$

$x^2 - y^2 = t^2 + 4 + \frac{4}{t^2} - \left(t^2 - 4 + \frac{4}{t^2}\right)$

$= 8$

5a.

$\int x\sqrt{x^2+3}\ dx$

$\frac{d}{dx}\ k(x^2+3)^{3/2} = x\,(x^2+3)^{1/2}$

$\frac{3}{2}k \cdot 2x \cdot (x^2+3)^{1/2} = x\,(x^2+3)^{1/2}$

$3kx = x$

$k = \frac{1}{3}$

$\therefore \int x\sqrt{x^2+3}\ dx = \frac{1}{3}(x^2+3)^{3/2} + c$

5b.

$\frac{dy}{dx} = \frac{x\sqrt{x^2+3}}{e^{2y}}$

$\int e^{2y}\,dy = \int x\sqrt{x^2+3}\ dx$

$\frac{1}{2}e^{2y} = \frac{1}{3}(x^2+3)^{3/2} + c$

when $y = 0$, $x = 1$

$\frac{1}{2} = \frac{1}{3}(4)^{3/2} + c$

$\frac{1}{2} = \frac{8}{3} + c \implies c = -\frac{13}{6}$

$$\frac{1}{2}e^{2y} = \frac{1}{3}(x^2+3)^{3/2} - \frac{13}{6} \qquad (\times 2)$$

$$e^{2y} = \frac{2}{3}(x^2+3)^{3/2} - \frac{13}{3}$$

$$2y = \ln\left\{\frac{2}{3}(x^2+3)^{3/2} - \frac{13}{3}\right\}$$

$$y = \frac{1}{2}\ln\left\{\frac{2}{3}(x^2+3)^{3/2} - \frac{13}{3}\right\}$$

6ai. $A(3,1,-6) \quad B(5,-2,0) \quad C(8,-4,-6)$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$$

$$= \begin{pmatrix} 8 \\ -4 \\ -6 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ -6 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ 0 \end{pmatrix} = 5\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

6aii.

$$\cos\theta = \frac{\overrightarrow{AC} \cdot \overrightarrow{BC}}{|AC|\,|BC|}$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$$

$$= \begin{pmatrix} 8 \\ -4 \\ -6 \end{pmatrix} - \begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ -6 \end{pmatrix}$$

$$\overrightarrow{AC} \cdot \overrightarrow{BC} = \begin{pmatrix} 5 \\ -5 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ -6 \end{pmatrix} = 5(3) + (-5)(-2) + 0(-6)$$

$$= 25$$

$$|AC| = \sqrt{5^2+5^2+0^2} = \sqrt{50} = 5\sqrt{2}$$

$$|BC| = \sqrt{3^2+2^2+6^2} = 7$$

$$\cos\theta = \frac{25}{\sqrt{50}\cdot 7} \quad \therefore \quad \theta = \cos^{-1}\left(\frac{5\sqrt{2}}{14}\right)$$

Rhombus iff $|AB| = |BC| = |CD| = |AD|$

$\vec{AB} = \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$, $|AB| = \sqrt{2^2+3^2+6^2} = 7$

$\vec{BC} = \begin{pmatrix} 3 \\ -2 \\ -6 \end{pmatrix}$, $|BC| = \sqrt{3^2+2^2+6^2} = 7$

$\vec{CD} = \begin{pmatrix} -2 \\ 3 \\ -6 \end{pmatrix}$, $|CD| = \sqrt{2^2+3^2+6^2} = 7$

$\vec{AD} = \begin{pmatrix} 3 \\ -2 \\ -6 \end{pmatrix}$, $|AD| = \sqrt{3^2+2^2+6^2} = 7$

$\therefore$ ABCD is a rhombus side length 7

7ai. $N = \dfrac{500}{1+9e^{-\frac{t}{8}}}$

when $t=0$, $N = \dfrac{500}{1+9e^0} = \dfrac{500}{\cancel{50}\ 10} = 50$

7aii. $t=24$, $N = \dfrac{500}{1+9e^{-\frac{24}{8}}} = 345$ (nearest whole number)

7aiii. $400 = \dfrac{500}{1+9e^{-\frac{t}{8}}}$

$1+9e^{-t/8} = \dfrac{500}{400}$

$9e^{-t/8} = \dfrac{1}{4}$

$e^{-t/8} = \dfrac{1}{36}$

$$-\frac{t}{8} = \ln\left(\frac{1}{36}\right)$$

$$t = -8\ln\left(\frac{1}{36}\right)$$

$$= 8\ln(36)$$

7bi.

$$N = 500\left(1+9e^{-t/8}\right)^{-1}$$

$$1+9e^{-t/8} = \frac{500}{N} \quad (*)$$

$$9e^{-t/8} = \frac{500}{N} - 1 \quad (+)$$

$$\frac{dN}{dt} = -500 \cdot -\frac{9}{8}e^{-t/8}\left(1+9e^{-t/8}\right)^{-2}$$

$$= \frac{-500}{-8} \cdot \underbrace{9e^{-t/8}}_{(+)} \underbrace{\left(1+9e^{-t/8}\right)}_{(*)}^{-2}$$

$$= \frac{125}{2} \cdot \left(\frac{500}{N} - 1\right)\left(\frac{500}{N}\right)^{-2}$$

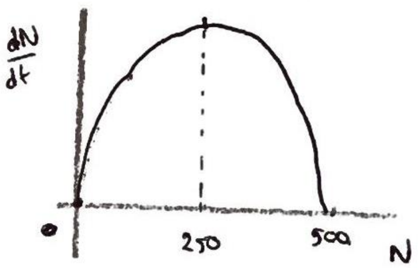
$$= \frac{125}{2} \cdot \left(\frac{500}{N} - 1\right) \cdot \frac{N^2}{500^2}$$

$$= \frac{N^2}{4000}\left(\frac{500}{N} - 1\right)$$

$$= \frac{N}{4000}\left(500 - N\right)$$

7bii.

$$\frac{dN}{dt} = \frac{N}{4000}(500 - N)$$



when $N = 0$, $\frac{dN}{dt} = 0$

$N = 500$, $\frac{dN}{dt} = 0$

so $\max \frac{dN}{dt} = 250 = N$

$$250 = \frac{500}{1+9e^{-t/8}}$$

$$1 + 9e^{-t/8} = 2$$

$$9e^{-t/8} = 1$$

$$e^{-t/8} = \frac{1}{9}$$

$$-\frac{t}{8} = \ln(1/9)$$

$$t = -8\ln(1/9)$$

$$= 8\ln 9$$

$$= 17.578 \quad (3dp)$$