AQA, Edexcel, OCR

A Level

A Level Mathematics

Proof by Contradiction

(Answers)

Name:

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Total Marks:

A1 – Proof Answers

AQA, Edexcel, OCR

1) **Prove that there is an infinite amount of prime numbers.**

Proof by contradiction.

[1 mark]

Assume there are a finite number of prime numbers, that we write as: **(1)**

$$p_1, p_2, p_3, \dots, p_n$$

[1 mark]

And we define a new number as

$$m = p_1 \times p_2 \times p_3 \times \dots \times p_n + 1$$

[1 mark]

As we are saying that there are no other prime numbers than the list defined in (1), then m should not be a prime number and therefore divisible by p_n.

[1 mark]

However, if we do this we are left with a remainder, 1, and as there are no integers that divide 1, then m must also be a prime number. This is the contradiction. Hence there are infinitely many prime numbers.

2) **For all real numbers if x^3 is rational, then x is also rational. True or false?**

[1 mark]

This is a true statement.

[1 mark]

Let x be a rational number, defined as

$$x = \frac{p}{q}$$

an irreducible fraction, where $p, q \in \mathbb{Z}$.

[1 mark]

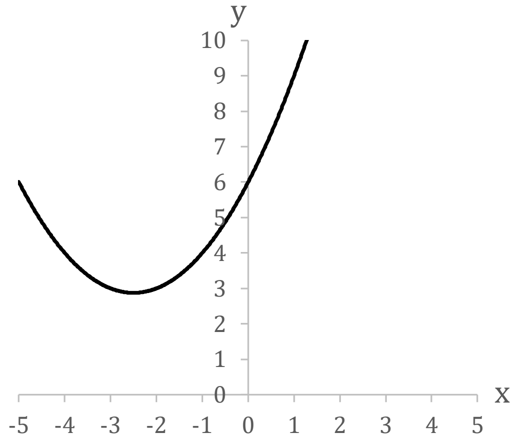
Cubing both sides of equation gives

$$x^3 = \frac{p^3}{q^3}$$

[1 mark]

We note that are integers because p and q are integers then so are their cubes. This means that x^3 is defined as the ratio of two integers, thus making it rational.

3)



The graph is defined as $kx^2 + 6kx + 5 = 0$ where k is constant. Prove that $0 \leq k \leq \frac{5}{9}$.

[1 mark]

Here you must spot that the graph does not intersect the *x-axis* and thus there are no real root solutions to this problem.

The graph clearly shows that the constant k is7 not negative.

[1 mark]

Insert $k = 0$, and show $0 + 0 + 5 = 0$ is not a viable solution.

[1 mark]

Note, using the quadratic equation discriminant that for non-real roots, $b^2 < 4ac$.

Inserting values of $a = k, b = 6k, c = 5$, gives

$$36k^2 < 20k$$

$$4k(9k - 5) < 0$$

$$0 < k < \frac{5}{9}$$

[1 mark]

However, we know $k = 0$, is a solution so we can modify it to:

$$0 \leq k < \frac{5}{9}$$

4) **Prove that $\sqrt{2}$ is irrational.**

Proof by contradiction.

[1 mark]

Assume that is rational and can be defined as

$$\sqrt{2} = \frac{a}{b}$$

an irreducible fraction, where $a, b \in \mathbb{Z}$.

[1 mark]

Squaring both sides gives

$$2 = \frac{a^2}{b^2}$$

$$2b^2 = a^2$$

[1 mark]

The LHS is an even number, this means that the RHS must also be an even number. Thus, both a and b are even.

[1 mark]

Contradiction. We originally stated that $\frac{a}{b}$ was irreducible, however if the integers were both even it would be reducible, by dividing by 2.

5) **If $\mathbf{a}, \mathbf{b} \in \mathbb{Z}$, then $\mathbf{a^2 - 4b - 3 \neq 0}$.**

Proof by contradiction.

[1 mark]

Assume the quadratic does equal zero. **(1)**

$$a^2 - 4b - 3 = 0$$

$$\Rightarrow a^2 = 4b + 3 \quad \textbf{(2)}$$

[1 mark]

The RHS here is odd, therefore, the LHS a^2 and ultimately a is odd. We can define a as

$$a = 2n + 1$$

[1 mark]

Substituting (2) back into (1) gives

$$(2n + 1)^2 = 4b + 3$$

$$4n^2 + 4n + 1 = 4b + 3$$

$$4(n^2 + n - b) = 2$$

$$(n^2 + n - b) = \frac{2}{4}$$

[1 mark]

Contradiction, on the LHS we have integers and on the RHS we have a fraction. Therefore, the assumption that the quadratic equals zero is incorrect.

6) **Using proof by contradiction show that there are no positive integer solutions to the Diophantine equation $x^2 - y^2 = 10$.**

[1 mark]

Assume positive integer solutions.

[1 mark]

Spot solution is difference of two squares. **(1)**

$$(x + y)(x - y) = 1 \quad \textbf{(2)}$$

$$x + y = 1, x - y = 1$$

$$x + y = -1, x - y = -1$$

Solving (1), by adding, gives:

$$x = 2, y = 0$$

[1 mark]

This is a contradiction as x and y should be positive.

Solving (2), by adding, gives:

$$x = -1, y = 0$$

[1 mark]

Again, this is a contradiction as x and y should be positive.

7) **If a is a rational number and b is an irrational number, then $a + b$ is an irrational number.**

Demonstrate, using proof, why the above statement is correct.

Proof by contradiction.

[1 mark]

Assume, a is a rational number, b is an irrational number $a + b$ is a rational number.

Therefore, a can be represented as the ratio of two integers,

$$\frac{m}{n}$$

b can be left the same and $a + b$ can also be represented as the ratio of two integers,

$$\frac{j}{k}$$

[1 mark]

Writing our assumptions out gives

$$\frac{m}{n} + b = \frac{j}{k}$$

$$\Rightarrow b = \frac{j}{k} - \frac{m}{n}$$

$$\Rightarrow b = \frac{km - nj}{kn}$$

[1 mark]

Contraction. This last statement says b equals the product of two integers (km) minus the product of two other integers (nj), all divided by another integer product (kn). This means b is rational. However, we know b is irrational so the assumption that rational + irrational = rational is incorrect.

8) **Prove that triangle ABC can have no more than one right angle.**

Proof by contradiction.

$$\angle A + \angle B + \angle C = 180^o$$

[1 mark]

If

$$\angle A = 90^0 and\ \angle B = 90^o$$

then

$$90^o + 90^o + \angle C = 180^o$$

$$\angle C = 0^o$$

[1 mark]

Contradiction. Triangles must have three angles, one cannot equal 0.

9) **Prove that the product of sum of three consecutive integers is divisible by 3.**

Let the first integer be n, the second $n+1$ and the third $n+2$.

[1 mark]

Their sum, therefore, is

$$n + (n+1) + (n+2)$$

$$3n + 3$$

$$3(n+1)$$

[1 mark]

And three is divisible by three.

10) **The number of even integers is limitless. Prove or disprove this statement.**

Proof by contradiction.

[1 mark]

Assume the number of even integers is limited and this largest number is called L.

$$L = 2n$$

as it is even.

[1 mark]

Consider, L+2

$$L + 2 = 2n + 2$$

$$L + 2 = 2(n+1)$$

which is also even and larger than L.

[1 mark]

This is a contradiction to our original assumption.

11) **Suppose a $\in \mathbb{Z}$ If $\boldsymbol{a^2}$ is even, then $\boldsymbol{a}$ is even.**

Proof by contradiction.

[1 mark]

Suppose $a^2$is not even, then we can define it as

$$a^2 = (2n+1)^2$$

$$a^2 = 4n^2 + 4n + 1$$

$$a^2 = 2(2n^2 + 2) + 1$$

which is an odd number.

[1 mark]

This means $a^2$is an odd number, if a is an even number, this makes a^2an even number too.

How can a^2 be both even and odd. It cannot.

12) **Prove that $\frac{d}{dx}\left(3^{\frac{1}{2}}x + \pi\right)$ is irrational.**

[1 mark]

Correctly differentiate the statement to give $3^{\frac{1}{2}}$, which is the same as $\sqrt{3}$.

Assume $\sqrt{3}$ is rational and can be represented as $\frac{m}{n}$, an irreducible fraction.

[1 mark]

$$\sqrt{3} = \frac{m}{n} \quad (1)$$

$$\Rightarrow 3 = \frac{m^2}{n^2} \quad (2)$$

$$\Rightarrow 3n^2 = m^2 \quad (3)$$

Assuming n is even, thus making m even, would mean that the original irreducible fraction $\frac{m}{n}$ could have been reduced. Assuming n is odd, this makes m also odd, allows us to continue with the proof.

[1 mark]

We can write

$$n = 2j + 1 \quad (4)$$

$$m = 2k + 1 \quad (5)$$

[1 mark]

Substituting (4) and (5) back into (3) gives

$$3(2j + 1)^2 = (2k + 1)^2$$

$$3(4j^2 + 4j + 1) = 4k^2 + 4k + 1$$

$$12j^2 + 12j + 2 = 4(k^2 + k)$$

$$6j^2 + 6j + 1 = 2(k^2 + k) \quad (6)$$

[1 mark]

Contradiction. On the left-hand side of (6) we have an odd integer (as we have two terms containing 6 plus 1) and on the right-hand side we have an even integer.

This means that our original assumption that $\sqrt{3}$ is rational is incorrect.