OCR
Oxford Cambridge and RSA

Tuesday 6 June 2023 – Afternoon

A Level Mathematics A

H240/01 Pure Mathematics

Time allowed: 2 hours

You must have:
- the Printed Answer Booklet
- a scientific or graphical calculator

QP

INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give non-exact numerical answers correct to **3** significant figures unless a different degree of accuracy is specified in the question.
- The acceleration due to gravity is denoted by $g\,\text{m}\,\text{s}^{-2}$. When a numerical value is needed use $g = 9.8$ unless a different value is specified in the question.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

INFORMATION
- The total mark for this paper is **100**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **12** pages.

ADVICE
- Read each question carefully before you start your answer.

Formulae
A Level Mathematics A (H240)

Arithmetic series

$S_n = \frac{1}{2}n(a+l) = \frac{1}{2}n\{2a+(n-1)d\}$

Geometric series

$S_n = \frac{a(1-r^n)}{1-r}$

$S_\infty = \frac{a}{1-r}$ for $|r| < 1$

Binomial series

$(a+b)^n = a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + \ldots + {}^nC_r a^{n-r}b^r + \ldots + b^n \qquad (n \in \mathbb{N})$

where ${}^nC_r = {}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$

$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots \qquad (|x| < 1, n \in \mathbb{R})$

Differentiation

$\mathrm{f}(x)$	$\mathrm{f}'(x)$
$\tan kx$	$k\sec^2 kx$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\mathrm{cosec}^2 x$
$\mathrm{cosec}\, x$	$-\mathrm{cosec}\, x \cot x$

Quotient rule $y = \frac{u}{v}$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$

Differentiation from first principles

$\mathrm{f}'(x) = \lim_{h \to 0} \frac{\mathrm{f}(x+h) - \mathrm{f}(x)}{h}$

Integration

$\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\,\mathrm{d}x = \ln|\mathrm{f}(x)| + c$

$\int \mathrm{f}'(x)(\mathrm{f}(x))^n\,\mathrm{d}x = \frac{1}{n+1}(\mathrm{f}(x))^{n+1} + c$

Integration by parts $\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$

Small angle approximations

$\sin\theta \approx \theta$, $\cos\theta \approx 1 - \frac{1}{2}\theta^2$, $\tan\theta \approx \theta$ where θ is measured in radians

Trigonometric identities

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \qquad \left(A \pm B \neq (k + \tfrac{1}{2})\pi\right)$

Numerical methods

Trapezium rule: $\int_a^b y\,\mathrm{d}x \approx \frac{1}{2}h\{(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\}$, where $h = \dfrac{b-a}{n}$

The Newton-Raphson iteration for solving $\mathrm{f}(x) = 0$: $x_{n+1} = x_n - \dfrac{\mathrm{f}(x_n)}{\mathrm{f}'(x_n)}$

Probability

$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B \mid A) = \mathrm{P}(B)\mathrm{P}(A \mid B)$ **or** $\mathrm{P}(A \mid B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$

Standard deviation

$\sqrt{\dfrac{\sum(x - \bar{x})^2}{n}} = \sqrt{\dfrac{\sum x^2}{n} - \bar{x}^2}$ **or** $\sqrt{\dfrac{\sum f(x - \bar{x})^2}{\sum f}} = \sqrt{\dfrac{\sum fx^2}{\sum f} - \bar{x}^2}$

The binomial distribution

If $X \sim \mathrm{B}(n, p)$ then $\mathrm{P}(X = x) = \dbinom{n}{x} p^x (1-p)^{n-x}$, mean of X is np, variance of X is $np(1-p)$

Hypothesis test for the mean of a normal distribution

If $X \sim \mathrm{N}(\mu, \sigma^2)$ then $\overline{X} \sim \mathrm{N}\left(\mu, \dfrac{\sigma^2}{n}\right)$ and $\dfrac{\overline{X} - \mu}{\sigma / \sqrt{n}} \sim \mathrm{N}(0, 1)$

Percentage points of the normal distribution

If Z has a normal distribution with mean 0 and variance 1 then, for each value of p, the table gives the value of z such that $\mathrm{P}(Z \leqslant z) = p$.

p	0.75	0.90	0.95	0.975	0.99	0.995	0.9975	0.999	0.9995
z	0.674	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291

Kinematics

Motion in a straight line

$v = u + at$

$s = ut + \frac{1}{2}at^2$

$s = \frac{1}{2}(u + v)t$

$v^2 = u^2 + 2as$

$s = vt - \frac{1}{2}at^2$

Motion in two dimensions

$\mathbf{v} = \mathbf{u} + \mathbf{a}t$

$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$

$\mathbf{s} = \frac{1}{2}(\mathbf{u} + \mathbf{v})t$

$\mathbf{s} = \mathbf{v}t - \frac{1}{2}\mathbf{a}t^2$

1 In the triangle ABC, the length $AB = 6\,\text{cm}$, the length $AC = 15\,\text{cm}$ and the angle $BAC = 30°$.

(a) Calculate the length BC. **[2]**

D is the point on AC such that the length $BD = 4\,\text{cm}$.

(b) Calculate the possible values of the angle ADB. **[3]**

2 **(a)** **(i)** Show that $\dfrac{1}{3-2\sqrt{x}} + \dfrac{1}{3+2\sqrt{x}}$ can be written in the form $\dfrac{a}{b+cx}$, where a, b and c are constants to be determined. **[2]**

(ii) Hence solve the equation $\dfrac{1}{3-2\sqrt{x}} + \dfrac{1}{3+2\sqrt{x}} = 2$. **[2]**

(b) **In this question you must show detailed reasoning.**

Solve the equation $2^{2y} - 7 \times 2^{y} - 8 = 0$. **[4]**

3 **(a)** Given that $f(x) = x^2 + 2x$, use differentiation from first principles to show that $f'(x) = 2x + 2$. **[4]**

(b) The gradient of a curve is given by $\dfrac{dy}{dx} = 2x + 2$ and the curve passes through the point $(-1, 5)$.

Find the equation of the curve. **[3]**

4 It is given that $ABCD$ is a quadrilateral. The position vector of A is $\mathbf{i} + \mathbf{j}$, and the position vector of B is $3\mathbf{i} + 5\mathbf{j}$.

(a) Find the length AB. **[1]**

(b) The position vector of C is $p\mathbf{i} + p\mathbf{j}$ where p is a constant greater than 1.

Given that the length AB is equal to the length BC, determine the position vector of C. **[3]**

(c) The point M is the midpoint of AC.

Given that $\overrightarrow{MD} = 2\overrightarrow{BM}$, determine the position vector of D. **[2]**

(d) State the name of the quadrilateral $ABCD$, giving a reason for your answer. **[2]**

5 **(a)** The function $f(x)$ is defined for all values of x as $f(x)=|ax-b|$, where a and b are positive constants.

(i) The graph of $y=f(x)+c$, where c is a constant, has a vertex at (3, 1) and crosses the y-axis at (0, 7).

Find the values of a, b and c. **[3]**

(ii) Explain why $f^{-1}(x)$ does not exist. **[1]**

(b) The function $g(x)$ is defined for $x \geqslant \frac{q}{p}$ as $g(x)=|px-q|$, where p and q are positive constants.

(i) Find, in terms of p and q, an expression for $g^{-1}(x)$, stating the domain of $g^{-1}(x)$. **[3]**

(ii) State the set of values of p for which the equation $g(x)=g^{-1}(x)$ has no solutions. **[1]**

6 A curve has equation $y=e^{x^2+3x}$.

(a) Determine the x-coordinates of any stationary points on the curve. **[4]**

(b) Show that the curve is convex for all values of x. **[5]**

7 **(a)** Use the result $\cos(A+B)=\cos A\cos B-\sin A\sin B$ to show that

$$\cos(A-B)=\cos A\cos B+\sin A\sin B.$$ **[2]**

The function $f(\theta)$ is defined as $\cos(\theta+30°)\cos(\theta-30°)$, where θ is in degrees.

(b) Show that $f(\theta)=\cos^2\theta-\frac{1}{4}$. **[3]**

(c) **(i)** Determine the following.

- The **maximum** value of $f(\theta)$
- The smallest **positive** value of θ for which this maximum value occurs **[2]**

(ii) Determine the following.

- The **minimum** value of $f(\theta)$
- The smallest **positive** value of θ for which this minimum value occurs **[2]**

8 **(a)** Find the first **three** terms in the expansion of $(4+3x)^{\frac{3}{2}}$ in ascending powers of x. **[4]**

(b) State the range of values of x for which the expansion in part **(a)** is valid. **[1]**

(c) In the expansion of $(4+3x)^{\frac{3}{2}}(1+ax)^2$ the coefficient of x^2 is $\frac{107}{16}$.

Determine the possible values of the constant a. **[4]**

9 Conservationists are studying how the number of bees in a wildflower meadow varies according to the number of wildflower plants. The study takes place over a series of weeks in the summer. A model is suggested for the number of bees, B, and the number of wildflower plants, F, at time t weeks after the start of the study.

In the model $B = 20+2t+\cos 3t$ and $F = 50\mathrm{e}^{0.1t}$.

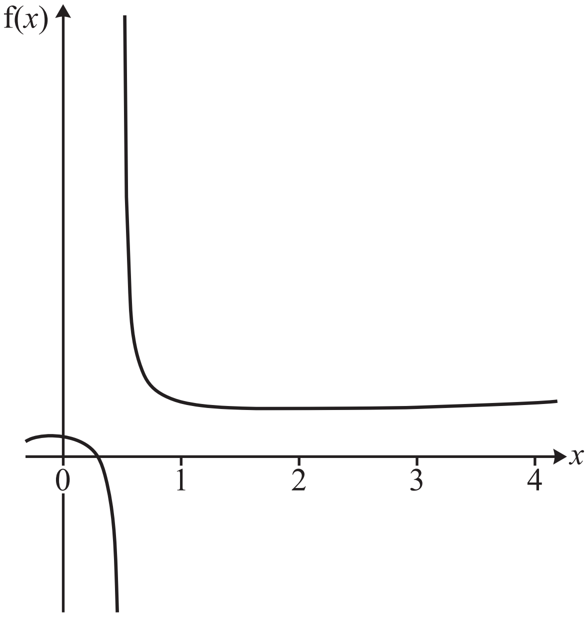
The model assumes that B and F can be treated as continuous variables.

(a) State the meaning of $\frac{\mathrm{d}B}{\mathrm{d}F}$. **[1]**

(b) Determine $\frac{\mathrm{d}B}{\mathrm{d}F}$ when $t = 4$. **[4]**

(c) Suggest a reason why this model may not be valid for values of t greater than 12. **[1]**

10



The diagram shows part of the curve $f(x) = \frac{e^x}{4x^2 - 1} + 2$. The equation $f(x) = 0$ has a positive root α close to $x = 0.3$.

(a) Explain why using the sign change method with $x = 0$ and $x = 1$ will fail to locate α. **[1]**

(b) Show that the equation $f(x) = 0$ can be written as $x = \frac{1}{4}\sqrt{(4 - 2e^x)}$. **[2]**

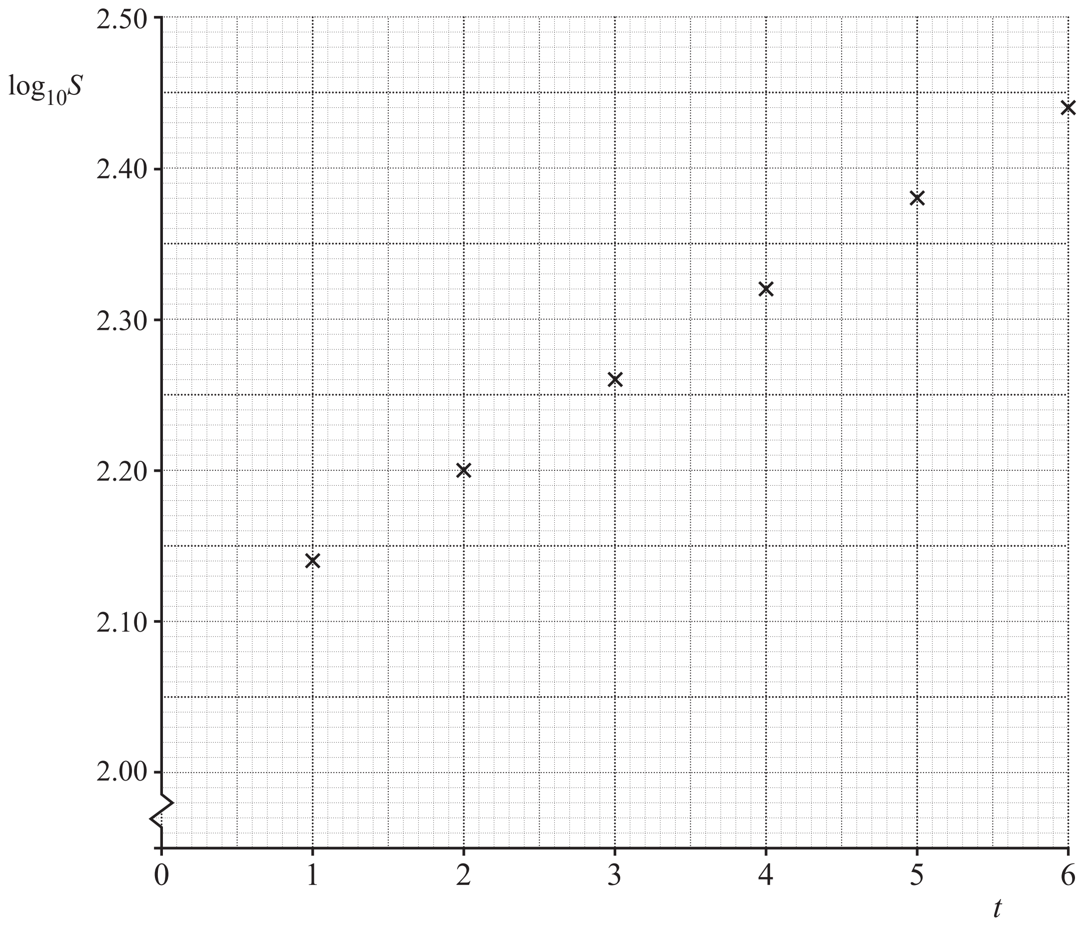
(c) Use the iterative formula $x_{n+1} = \frac{1}{4}\sqrt{(4 - 2e^{x_n})}$ with a starting value of $x_1 = 0.3$ to find the value of α correct to **4** significant figures, showing the result of each iteration. **[3]**

(d) An alternative iterative formula is $x_{n+1} = F(x_n)$, where $F(x_n) = \ln(2 - 8x_n^{\,2})$.

By considering $F'(0.3)$ explain why this iterative formula will not find α. **[3]**

11 The owners of an online shop believe that their sales can be modelled by $S = ab^t$, where a and b are both positive constants, S is the number of items sold in a month and t is the number of complete months since starting their online shop.

The sales for the first six months are recorded, and the values of $\log_{10}S$ are plotted against t in the graph below. The graph is repeated in the Printed Answer Booklet.



(a) Explain why the graph suggests that the given model is appropriate. **[3]**

The owners believe that $a = 120$ and $b = 1.15$ are good estimates for the parameters in the model.

(b) Show that the graph supports these estimates for the parameters. **[2]**

(c) Use the model $S = 120 \times 1.15^t$ to predict the number of items sold in the **seventh** month after opening. **[2]**

(d) **(i)** Use the model $S = 120 \times 1.15^t$ to predict the number of months after opening when the **total** number of items sold after opening will first exceed 70 000. **[4]**

(ii) Comment on how reliable this prediction may be. **[1]**

12 **(a)** Use the substitution $u = e^x - 2$ to show that

$$\int \frac{7e^x - 8}{(e^x - 2)^2}\,dx = \int \frac{7u + 6}{u^2(u + 2)}\,du.$$

[3]

(b) Hence show that

$$\int_{\ln 4}^{\ln 6} \frac{7e^x - 8}{(e^x - 2)^2}\,dx = a + \ln b$$

where a and b are rational numbers to be determined. **[7]**

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