Oxford Cambridge and RSA

Tuesday 21 June 2022 – Afternoon

A Level Mathematics B (MEI)

H640/03 Pure Mathematics and Comprehension

Insert

Time allowed: 2 hours

9122294576

INSTRUCTIONS
- Do **not** send this Insert for marking. Keep it in the centre or recycle it.

INFORMATION
- This Insert contains the article for Section B.
- This document has **4** pages.

Approximating the sine function

Small angles

For a small angle x radians, the approximation $\sin x \approx x$ is valid. The curve $y = \sin x$ and the straight line $y = x$ are shown in **Fig. C1.1**. **Fig. C1.2** shows the curve $y = x - \sin x$. Inspection of the graphs suggests that x is a reasonable approximation for $\sin x$ for $-0.5 \leqslant x \leqslant 0.5$ and also that $y = x$ has the same gradient as $y = \sin x$ when $x = 0$.

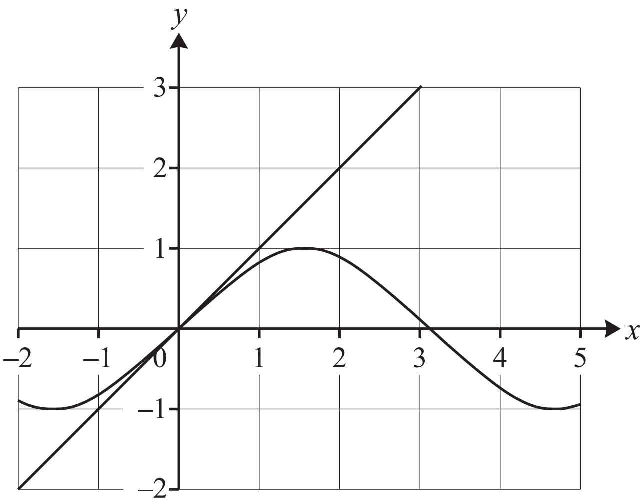

Fig. C1.1

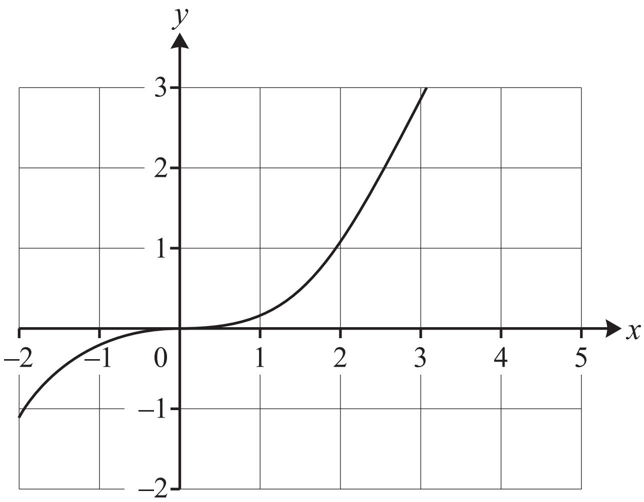

Fig. C1.2

Calculating sin *x*

Trigonometric functions, including $\sin x$, are widely used so it is useful to be able to calculate the value of the sine of any angle accurately and quickly. This is easily done nowadays using a calculator but this was not possible in the past. The linear function, $y = x$, is only a reasonable approximation for $y = \sin x$ for values of x close to zero. Perhaps using a higher degree polynomial would give a reasonable approximation for a wider range of values of x.

Fig. C2.1 shows the curve $y = \sin x$ and the quadratic curve which goes through the points (0, 0), $\left(\frac{\pi}{2}, 1\right)$ and $(\pi, 0)$. The equation of this curve is $y = \frac{4x(\pi - x)}{\pi^2}$. **Fig. C2.2** shows the curve $y = \frac{4x(\pi - x)}{\pi^2} - \sin x$.

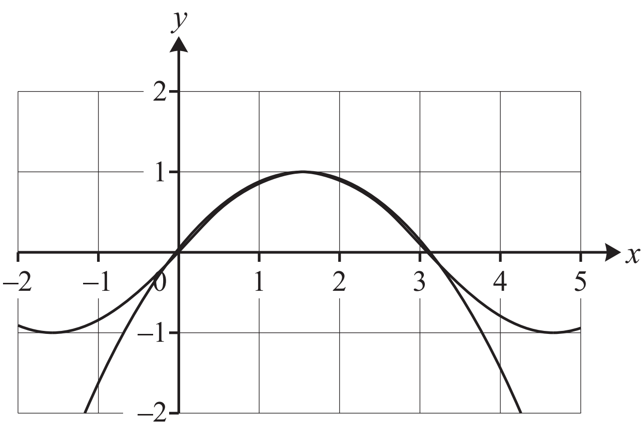

Fig. C2.1

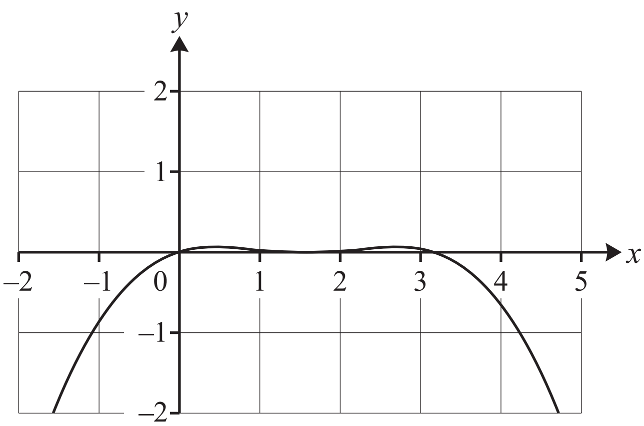

Fig. C2.2

The quadratic function seems to be a reasonably good approximation for $\sin x$ in the interval $0 \leqslant x \leqslant \pi$. However, calculating percentage errors for selected values of x shows that the percentage errors made by using the quadratic function as an approximation to $\sin x$ are quite high for values of x close to zero or π.

The spreadsheet in **Fig. C3** shows values of x in column A, with the corresponding values of $\sin x$ and the quadratic function $\dfrac{4x(\pi - x)}{\pi^2}$ in columns B and C. Columns D and E show the percentage errors in using x and the quadratic as approximations for $\sin x$.

	A	B	C	D	E
1	x	sin(x)	quadratic	% error for x	% error for quadratic
2	0	0	0		
3	0.1	0.099833	0.123271	0.166861	23.476799
4	0.2	0.198669	0.238437	0.669791	20.016773
5	0.3	0.295520	0.345496	1.515901	16.911206
6	0.4	0.389418	0.444450	2.717298	14.131825

Fig. C3

A better approximation

The approximation $\sin x \approx \dfrac{16x(\pi - x)}{5\pi^2 - 4x(\pi - x)}$ was discovered by an Indian mathematician named Bhaskara in the 7th century. It is not known how Bhaskara derived the formula but it can be seen that the curve $y = \dfrac{16x(\pi - x)}{5\pi^2 - 4x(\pi - x)}$ is symmetrical about $x = \dfrac{\pi}{2}$ and goes through the points (0, 0), $\left(\dfrac{\pi}{2}, 1\right)$ and $(\pi, 0)$. **Fig. C4** shows the curves $y = \sin x$ and $y = \dfrac{16x(\pi - x)}{5\pi^2 - 4x(\pi - x)}$. Radians were not in use until the 18th century; Bhaskara gave the formula for an angle θ degrees as $\sin\theta \approx \dfrac{4\theta(180 - \theta)}{40500 - \theta(180 - \theta)}$.

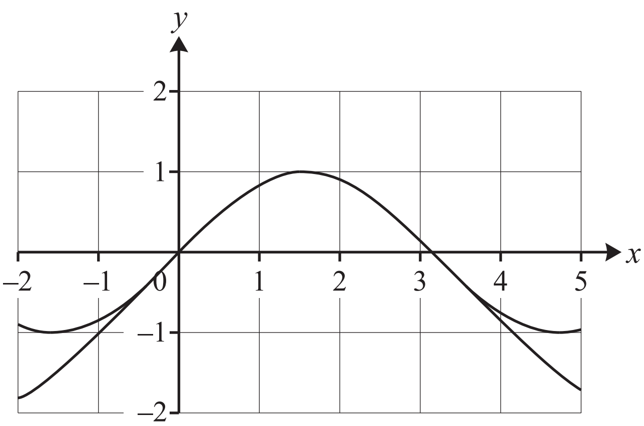

Fig. C4

The percentage error in approximating $\sin x$ by $\dfrac{16x(\pi - x)}{5\pi^2 - 4x(\pi - x)}$ is less than 2% throughout the interval $0 \leqslant x \leqslant \pi$. The Bhaskara approximation for $\sin x$ can be used to derive the following approximation for $\cos x$; $\cos x \approx \dfrac{\pi^2 - 4x^2}{\pi^2 + x^2}$.