OCR
Oxford Cambridge and RSA

Thursday 23 June 2022 – Afternoon

A Level Further Mathematics A

Y545/01 Additional Pure Mathematics

Time allowed: 1 hour 30 minutes

You must have:
- the Printed Answer Booklet
- the Formulae Booklet for A Level Further Mathematics A
- a scientific or graphical calculator

9460359963

INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give non-exact numerical answers correct to **3** significant figures unless a different degree of accuracy is specified in the question.
- The acceleration due to gravity is denoted by $g\,\text{m}\,\text{s}^{-2}$. When a numerical value is needed use $g = 9.8$ unless a different value is specified in the question.
- Do **not** send this Question Paper for marking. Keep in the centre or recycle it.

INFORMATION
- The total mark for this paper is **75**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **8** pages.

ADVICE
- Read each question carefully before you start your answer.

Answer **all** the questions.

1 The surface E has equation $z = \sqrt{500 - 3x^2 - 2y^2}$.

(a) Determine the values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the point P on E with coordinates $(11, -8, 3)$. **[4]**

(b) Find the equation of the tangent plane to E at P, giving your answer in the form $ax + by + cz = d$ where a, b, c and d are integers. **[2]**

2 Consider the integers a and b, where, for each integer n, $a = 7n + 4$ and $b = 8n + 5$.

Let $h = \mathrm{hcf}(a, b)$.

(a) Determine all possible values of h. **[3]**

(b) Find all values of n for which a and b are **not** co-prime. **[2]**

3 The irrational number $\phi = \frac{1}{2}(1 + \sqrt{5})$ plays a significant role in the sequence of Fibonacci numbers given by $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geqslant 1$.

Prove by induction that, for each positive integer n, $\phi^n = F_n \times \phi + F_{n-1}$. **[6]**

4 Let N be the number $15\,824\,578$.

(a) **(i)** Use a standard divisibility test to show that N is a multiple of 11. **[2]**

(ii) A student uses the following test for divisibility by 7.

> 'Throw away' multiples of 7 that appear either individually or within a pair of consecutive digits of the test number.
> Stop when the number obtained is 0, 1, 2, 3, 4, 5 or 6.
> The test number is only divisible by 7 if that obtained number is 0.

For example, for the number N, they first 'throw away' the "7" in the tens column, leaving the number $N_1 = 15824508$. At the second stage, they 'throw away' the "14" from the left-hand pair of digits of N_1, leaving $N_2 = 01824508$; and so on, until a number is obtained which is 0, 1, 2, 3, 4, 5 or 6.

- Justify the validity of this process.
- Continue the student's test to show that $7 \mid N$. **[2]**

(iii) Given that $N = 11 \times 1\,438\,598$, explain why $7 \mid 1\,438\,598$. **[1]**

(b) Let $M = N^2$.

(i) Express N in the unique form $101a + b$ for positive integers a and b, with $0 \leqslant b < 101$. **[2]**

(ii) Hence write M in the form $M \equiv r \pmod{101}$, where $0 < r < 101$. **[1]**

(iii) Deduce the order of N modulo 101. **[1]**

5 You are given the variable point $A(3, -8, t)$, where t is a real parameter, and the fixed point $B(1, 2, -2)$.

(a) Using only the geometrical properties of the vector product, explain why the statement "$\overrightarrow{OA} \times \overrightarrow{OB} = \mathbf{0}$" is false for all values of t. **[2]**

(b) **(i)** Use the vector product to find an expression, in terms of t, for the area of triangle OAB. **[4]**

(ii) Hence determine the value of t for which the area of triangle OAB is a minimum. **[2]**

6 In a national park, the number of adults of a given species is carefully monitored and controlled. The number of adults, n months after the start of this project, is A_n. Initially, there are 1000 adults. It is predicted that this number will have declined to 960 after one month.

The first model for the number of adults is that, from one month to the next, a fixed proportion of adults is lost. In order to maintain a fixed number of adults, the park managers "top up" the numbers by adding a constant number of adults from other parks at the end of each month.

(a) Use this model to express the number of adults as a first-order recurrence system. **[1]**

Instead, it is found that, the proportion of adults lost each month is double the predicted amount, with no change being made to the constant number of adults added each month.

(b) (i) Show that the revised recurrence system for A_n is $A_0 = 1000,\ A_{n+1} = 0.92A_n + 40$. **[1]**

(ii) Solve this revised recurrence system. **[4]**

(iii) Describe the long-term behaviour of the sequence $\{A_n\}$ in this case. **[1]**

A more refined model for the number of adults uses the second-order recurrence system $A_{n+1} = 0.9A_n - 0.1A_{n-1} + 50$, for $n \geqslant 1$, with $A_0 = 1000$ and $A_1 = 920$.

(c) (i) Determine the long-term behaviour of the sequence $\{A_n\}$ for this more refined model. **[4]**

(ii) A criticism of this more refined model is that it does not take account of the fact that the number of adults must be an integer at all times.

State a modified form of the second-order recurrence relation for this more refined model that will satisfy this requirement. **[1]**

7 **(a)** Differentiate $(16+t^2)^{\frac{3}{2}}$ with respect to t. **[1]**

Let $I_n = \int_0^3 t^n \sqrt{16+t^2}\,dt$ for integers $n \geqslant 1$.

(b) Show that, for $n \geqslant 3$, $(n+2)I_n = 125 \times 3^{n-1} - 16(n-1)I_{n-2}$. **[5]**

(c) The curve C is defined parametrically by $x = t^4 \cos t,\ y = t^4 \sin t$, for $0 \leqslant t \leqslant 3$. The length of C is denoted by L.

Show that $L = I_3$. (You are not required to evaluate this integral.) **[4]**

8 **(a)** Explain why all groups of even order must contain at least one self-inverse element (that is, an element of order 2). **[2]**

(b) Prove that any group, in which every (non-identity) element is self-inverse, is abelian. **[2]**

(c) A student believes that, if x and y are two distinct, non-identity, self-inverse elements of a group, then the element xy is also self-inverse.

The table shown here is the Cayley table for the non-cyclic group of order 6, having elements i, a, b, c, d and e, where i is the identity.

	i	a	b	c	d	e
i	i	a	b	c	d	e
a	a	i	d	e	b	c
b	b	e	i	d	c	a
c	c	d	e	i	a	b
d	d	c	a	b	e	i
e	e	b	c	a	i	d

By considering the elements of this group, produce a counter-example which proves that this student is wrong. **[2]**

(d) A group G has order $4n+2$, for some positive integer n, and i is the identity element of G. Let x and y be two distinct, non-identity, self-inverse elements of G. By considering the set $H=\{i, x, y, xy\}$, prove by contradiction that not all elements of G are self-inverse. **[4]**

9 For all real values of x and y the surface S has equation $z=4x^2+4xy+y^2+6x+3y+k$, where k is a constant and an integer.

(a) Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$. **[2]**

(b) Determine the smallest value of the integer k for which the whole of S lies above the x-y plane. **[7]**

END OF QUESTION PAPER

BLANK PAGE

BLANK PAGE